TOPOLOGY SPRING 2024
SERIE 11

(1) A topological space X is called *path-connected* if, for every x and y in X, there exists a path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.
   (a) Show that if X is path-connected, then it is connected.
   (b) Show that the relation $\sim$ defined by $x \sim y$ if and only if there exists a path in X from x to y is an equivalence relation.
   (c) Show that the equivalence class of some $x \in X$ for the relation $\sim$ is contained in the connected component of x in X. This equivalence class is called the *path-connected component* of x.
   (d) We now assume that X is a connected topological manifold. Show that the path-connected component of any $x \in X$ is open.
   (e) Let X be a contractible space. Show that X is path connected. (Hint: if Id_X is homotopic to the constant x_0, show first that for all x, there is a path in X from x to x_0.)

(2) Let X be the subspace

$$C = \{(0,1)\} \ \cup \ \bigcup_{n\geq 1}(\{1/n\} \times [0,1]) \ \cup \ ([0,1] \times \{0\}) \subset \mathbf{R}^2$$

of the plane, with the induced topology.
   (a) Prove that C is connected. (Hint: sketch C.)
   (b) Let $\gamma\colon [0,1] \to C$ be a path with $\gamma(0) = (0,1)$ and let $Y = \gamma^{-1}(\{(0,1)\})$. Show that $Y \subset [0,1]$ is closed and not empty.
   (c) Let $t_0 \in Y$. Let $\varepsilon > 0$ be a real number with $\varepsilon < 1/2$ and let

$$V = \{(x,y) \in C \mid |x| + |y-1| < \varepsilon\}.$$

   Show that there exist real numbers a, b with $0 < a < t_0 < b < 1$ such that $\gamma(]a,b[) \subset V$.
   (d) Show that $\gamma(]a,b[) \subset V$ is connected.
   (e) Deduce that $\gamma(]a,b[) = \{(0,1)\}$. (Hint: note first that $\gamma(]a,b[)$ does not intersect the real axis; show further that $\gamma(]a,b[)$ cannot contain an element of the form $(1/n, u)$ for some $n \geq 1$, using the previous question.)
   (f) Deduce that Y is open in $[0,1]$.
   (g) Conclude that C is not path-connected.

(3) Let X and Y be topological spaces. A continuous map $f\colon X \to Y$ is called a *homotopy equivalence* if there exists a continuous map $g\colon Y \to X$ such that $f \circ g$ is homotopic to Id_Y and $g \circ f$ is homotopic to Id_X. If there exists a homotopy equivalence from X to Y, then X and Y are said to have the same *homotopy type*.
   (a) Show that the relation "X has the same homotopy type as Y" is an equivalence relation on topological spaces.

(b) If $f\colon X \to Y$ is a homotopy equivalence, show that the homotopy class in $[Y, X]$ of a map $g\colon Y \to X$ such that $f \circ g \sim \mathrm{Id}_Y$ and $g \circ f \sim \mathrm{Id}_X$ is unique. This class is called the *homotopy inverse* of f.

(c) Show that if $f\colon X \to Y$ is a homotopy equivalence with homotopy inverse g and $f'\colon Y \to Z$ is a homotopy equivalence with homotopy inverse g', then $f' \circ f$ is a homotopy equivalence, with homotopy inverse $g \circ g'$.

(d) Show that X has the same homotopy type as a one-point space $\{x_0\}$ if and only if X is contractible.

(e) Show that if X has the same homotopy type as a point, then for any space Y, any continuous map $Y \to X$ is homotopic to a constant map, and any continuous map $X \to Y$ is homotopic to a constant map.

(4) A subspace Y of a topological space X is called a *retract* of X if there exists a continuous map $r\colon X \to Y$ such that $r(y) = y$ for all $y \in Y$.

(a) If Y is a retract of X and $y_0 \in Y$, show that the group morphism

$$\pi_1(Y, y_0) \to \pi_1(X, y_0)$$

induced by the inclusion $Y \to X$ is injective.

(b) For $n \geq 1$, show that

$$\mathbf{S}_{n-1} = \{x \in \mathbf{R}^n \mid \|x\| = 1\}$$

is a retract of $\mathbf{R}^n \setminus \{0\}$.

(c) Show that the fundamental group of $\mathbf{R}^2 \setminus \{0\}$ is not trivial.

(d) Show that $\mathbf{S}_1$ is not a retract of $\mathbf{R}^2$. (Hint: show that this would imply that $\mathbf{S}_1$ is contractible.)

(5) Let $D = \{z \in \mathbf{C} \mid |z| \leq 1\}$ be the unit disc in $\mathbf{C}$. Let $f\colon D \to D$ be a continuous map. We assume that $f(z) \neq z$ for all $z \in D$.

(a) Show that there is a well-defined map $g\colon D \to \mathbf{S}_1$ which maps z to the intersection point of the line joining z and $f(z)$ with $\mathbf{S}_1$.

(b) Show that g is continuous.

(c) Deduce a contradiction and conclude that there must exist a fixed point of f. (Hint: use the previous exercise.)

The result of this exercise is *Brouwer's fixed-point theorem*, in dimension 2 (it is also valid for the unit ball in $\mathbf{R}^n$ for $n \geq 3$).