TOPOLOGY SPRING 2024
SERIE 12

(1) Let $X \subset \mathbf{R}^2$ be the union of the circles C_n with radius $1/n$ centered at $(1/n, 0)$ for all $n \geq 1$. Each of them passes by the origin $(0,0)$, and we let $x_0 = (0,0) \in X$. The goal of this exercise is to prove that $\pi_1(X, x_0)$ is an *uncountable* group (where X has the subspace topology from $\mathbf{R}^2$).

(a) Show that X is path-connected (an intuitive explanation is enough).

(b) Show that if U is a neighborhood of x_0, then there exists N such that $C_n \subset U$ for all $n \geq N$.

(c) Let $n \geq 1$ be an integer. Show that the map $r_n \colon X \to C_n$ such that

$$r_n(x) = \begin{cases} x & \text{if } x \in C_n, \\ x_0 & \text{if } x \notin C_n \end{cases}$$

is continuous. (Hint: since X is a metric space, you can use sequences here.)

(d) Show that the induced map $r_{n*} \colon \pi_1(X, x_0) \to \pi_1(C_n, x_0)$ is surjective.

(e) For $n \geq 1$, let $\gamma_n \colon [0,1] \to C_n$ be a loop at x_0 on C_n. Define $\gamma \colon [0,1] \to X$ by

$$\gamma(t) = \gamma_n\left(n(n+1)(t - 1 + \frac{1}{n})\right) \text{ if } n \text{ is such that } 1 - \frac{1}{n} \leq t < 1 - \frac{1}{n+1}$$

and $\gamma(1) = x_0$. Show that γ is a well-defined continuous loop at x_0. (Hint: for continuity, Question (b) will be useful.)

(f) Show that the class of $r_{n*}(\gamma)$ in $\pi_1(C_n, x_0)$ is the class of γ_n in $\pi_1(C_n, x_0)$ for all $n \geq 1$.

(g) Conclude that there is a surjective group morphism

$$\pi_1(X, x_0) \to \prod_n \pi_1(C_n, x_0),$$

and that $\pi_1(X, x_0)$ is uncountable.

(2) Let X be a topological space and $x_0 \in X$. Let $(U_i)_{i \in I}$ be open sets in X, all containing x_0, such that X is the union of the U_i's and $U_i \cap U_j$ is path-connected for all i and j in I.

(a) Let $\gamma \colon [0,1] \to X$ be a loop at x_0. Show that there exists an integer $m \geq 1$ and real numbers

$$t_0 = 0 < t_1 < \cdots < t_{m-1} < t_m = 1$$

such that for $0 \leq k < m$, the subset $\gamma([t_k, t_{k+1}])$ is contained in $U_{i(k)}$ for some $i(k) \in I$.

(b) Show that there exist loops γ_k at x_0 for $1 \leq k \leq m$ such that

$$\gamma \sim_p \gamma_1 \cdots \gamma_m$$

and moreover $\gamma_k([0,1]) \subset U_{i(k)}$, where $\sim_p$ is the relation of path-homotopy. (Hint: a picture, in the case where I has two elements, will help constructing the γ_k's.)

(c) If $\pi_1(U_i, x_0) = \{\varepsilon_{x_0}\}$ for all i, deduce that $\pi_1(X, x_0) = \{\varepsilon_{x_0}\}$.

(3) Let X be a topological space. Let $(A_i)_{i \in I}$ be subsets of X which are path-connected and such that

$$\bigcap_{i \in I} A_i$$

is not empty. Prove that

$$\bigcup_{i \in I} A_i$$

is path-connected.

(4) Let

$$\mathbf{S}_2 = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

Let $p = (1, 0, 0)$ and $q = (-1, 0, 0)$ in $\mathbf{S}_2$.

(a) Show that $\mathbf{S}_2$ and $\mathbf{S}_2 \setminus \{p, q\}$ are path-connected. (Hint: there are many different solutions; for instance you can use the previous exercise, or describe explicit paths joining two points.)

(b) Let $x_0 = (0, 1, 0)$. Show that $\pi_1(\mathbf{S}_2 \setminus \{p\}, x_0)$ and $\pi_1(\mathbf{S}_2 \setminus \{q\}, x_0)$ are both trivial groups.

(c) Deduce that $\pi_1(\mathbf{S}_2, x) = \{\varepsilon_x\}$ for all $x \in \mathbf{S}_2$.