TOPOLOGY SPRING 2024
SERIE 13

(1) Let $p\colon Y \to X$ be a covering space.
   (a) Show that for any $y \in Y$, there exists an open neighborhood V of y such that the restriction of p to V defines a homeomorphism $V \to p(V)$. (One says that a covering space is a *local homeomorphism*.)
   (b) Assume that X is connected and that $p^{-1}(\{x\})$ is finite for all $x \in X$. Prove then the cardinality of $p^{-1}(\{x\})$ is the same for all $x \in X$.

(2) Let $n \geq 1$ be an integer.
   (a) Show that the symmetric group S_n acts on $\mathbf{C}^n$ by
$$\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$
   for all $\sigma \in S_n$ of $(x_i) \in \mathbf{C}^n$.
   (b) Show that the projection map $p\colon \mathbf{C}^n \to \mathbf{C}^n/S_n$ is *not* a covering space if $n \geq 2$, but is one if $n = 1$. (Hint: one can for instance use the previous exercise.)
   (c) Let
$$U_n = \{(x_1, \ldots, x_n) \in \mathbf{C}^n \mid x_i \neq x_j \text{ if } i \neq j\}.$$
   Show that $\sigma \cdot U_n = U_n$ for all $\sigma \in S_n$, and that the projection map $U_n \to U_n/S_n$ is a covering space.

(3) Let $f \in \mathbf{C}[X]$ be a polynomial of degree $d \geq 1$. We consider it as a continuous map $f\colon \mathbf{C} \to \mathbf{C}$. Let Z_f denote the set of zeros of f', and $C_f = f(Z_f)$ the set of *critical values* of f.
   (a) Show that if $d \geq 2$, then f is *not* a covering space.
   (b) Let
$$U_f = \{z \in \mathbf{C} \mid f'(z) \neq 0\}.$$
   Show that for any $z \in U_f$, there exists an open neighborhood V_z of z such that the restriction of f to V_z defines a homeomorphism $V_z \to f(V_z)$. (Hint: show that there is a *compact* neighborhood of z on which f is injective.)
   (c) Show that the map f defines a covering space
$$U_f \to \mathbf{C} \setminus C_f = \{z \in \mathbf{C} \mid z \notin C_f\}.$$
   (Hint: given $w_0 \in \mathbf{C} \setminus C_f$, show that $f^{-1}(\{w_0\})$ has size d, and apply the previous question to each $z_0 \in f^{-1}(\{w_0\})$; then, construct an open neighborhood of w_0 over which f is trivializable by suitably exploiting the neighborhoods V_{z_0}.)

(4) Let $p\colon Y \to X$ be a covering space. Let $g\colon X' \to X$ be a continuous map. Define
$$Y' = \{(x, y) \in X' \times Y \mid g(x) = p(y)\},$$
with the subspace topology of the product topology. Let $p_1\colon Y' \to X'$ be the projection $p_1(x, y) = x$ and $p_2\colon Y' \to Y$ be the projection $p_2(x, y) = y$.

(a) Show that p_1 and p_2 are continuous and satisfy $g \circ p_1 = p \circ p_2$.

(b) Suppose that p is a trivial covering with $Y = X \times D$ for some non-empty discrete space D. Show that the map

$$\Psi \colon Y' \to X' \times D$$

defined by $\Psi(x, (v, d)) = (x, d)$ for $(x, (v, d)) \in X' \times (X \times D)$ is a homeomorphism, where $X' \times D$ has the product topology. (Hint: describe the reciprocal bijection.)

(c) Deduce that, in the general case, the projection $p_1 \colon Y' \to X'$ is a covering space; it is called the *pullback of p along g*.

(d) Show for every $x \in X'$, there is a bijection $p_1^{-1}(\{x\}) \to p^{-1}(\{g(x)\})$.

(5) For any integer $n \geq 1$, let $f_n \colon \mathbf{S}_1 \to \mathbf{S}_1$ be the covering space defined by $f_n(z) = z^n$.

(a) Show that the pullback of f_n along f_n (defined in the previous exercise) is a trivial covering of degree n.

(b) Let $p_1 \colon Y' \to \mathbf{S}_1$ be the pullback of f_2 along f_3. Show that the map

$$q \colon \mathbf{S}_1 \to Y'$$

defined by $q(z) = (z^2, z^3)$ is a homeomorphism such that $p_1 \circ q = f_2$. (This means that the pullback of f_2 along f_3 is "isomorphic" to f_2, as a covering space of $\mathbf{S}_1$.)