TOPOLOGY SPRING 2024
SERIE 14

Note: this exercise won't be graded.

(1) Let $f\colon Y \to X$ be a covering space and let $g\colon Z \to X$ be a continuous map. Let $z_0 \in Z$ and define $x_0 = g(Z_0)$. Let $y_0 \in Y$ be any element such that $f(y_0) = x_0$.

(a) Assume that g admits a lift $\tilde{g}\colon Z \to Y$ (i.e., we have $f \circ \tilde{g} = g$) such that $g(z_0) = y_0$. Show that

(1) $$g_*(\pi_1(Z, z_0)) \subset f_*(\pi_1(Y, y_0)) \subset \pi_1(X, x_0).$$

The goal of the reminder of this exercise is to prove the converse of this result when Z is path connected and locally path connected. Thus, we do not assume that g has a lift, but we assume that Z has these properties and that

$$g_*(\pi_1(Z, z_0)) \subset f_*(\pi_1(Y, y_0)) \subset \pi_1(X, x_0).$$

(b) For any $z \in Z$ and any path $\gamma\colon [0,1] \to Z$ from z_0 to z, show that there exists a unique lift $\eta\colon [0,1] \to Y$ of $g \circ \gamma$ such that $\eta(0) = y_0$.

(c) Let γ' be another path in Z from z_0 to z, and η' the corresponding lift of $g \circ \gamma'$ to Y such that $\eta'(0) = y_0$. Show that there exists a loop $\alpha\colon [0,1] \to Y$ at y_0 and a homotopy $h\colon [0,1] \times [0,1] \to X$ such that

$$h(s,0) = (g \circ \gamma') * (g \circ \gamma)(s), \qquad h(s,1) = (f \circ \alpha)(s).$$

(Hint: use equation (1).)

(d) Show that h admits a lift $\tilde{h}\colon [0,1] \times [0,1] \to Y$ such that $\tilde{h}(s,1) = \alpha(s)$.

(e) Deduce that $\tilde{h}_0\colon s \mapsto \tilde{h}(s,0)$ is a loop at y_0.

(f) Show that

$$\tilde{h}_0(s) = \begin{cases} \eta(2s) & \text{for } 0 \le s \le 1/2, \\ \eta'(2s-1) & \text{for } 1/2 \le s \le 1. \end{cases}$$

(Hint: use the uniqueness properties of the homotopy-lifting theorem.)

(g) Deduce that $\eta(1) = \eta'(1)$ and that the map $\tilde{g}\colon Z \to Y$ such that $\tilde{g}(z) = \eta(1)$ is well-defined.

(h) Let $z \in Z$ and let U be a neighborhood of $g(z)$ in X such that f is trivializable over U. Show that there exists a neighborhood $\tilde{U}$ of $\tilde{g}(z)$ such that the restriction $f_{\tilde{U}}$ of f to $\tilde{U}$ is a homeomorphism $f_{\tilde{U}}\colon \tilde{U} \to U$.

(i) Show that there exists a path connected neighborhood V of z such that $g(V) \subset U$; show then that for $w \in V$, we have $\tilde{g}(w) = f^{-1}_{g(V)}(g(w))$, and deduce that $\tilde{g}$ is continuous. (Hint: write a path γ from z_0 to $w \in V$ as $\gamma_0 * \gamma_w$ where γ_0 is a fixed path from z_0 to z and γ_w is a path from z to w, and find an explicit lift of γ_w using $f_{g(V)}$.)

(j) Show that $f \circ \tilde{g} = g$ and $\tilde{g}(z_0) = y_0$, hence $\tilde{g}$ is a lift of g mapping z_0 to y_0.