TOPOLOGY SPRING 2024
SERIE 2

(1) Let X be a set.

  (a) Let d_1 and d_2 be distances on X. If there exist $a, b > 0$ such that
$$d_1(x, y) \leq d_2(x, y)^a$$
and
$$d_2(x, y) \leq d_1(x, y)^b$$
for all $(x, y) \in X \times X$, show that the topologies defined by d_1 and d_2 are the same;

  (b) Let d be a distance on X. Show that
$$\delta(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$
is a distance on X. Show that the topology defined by δ is the same as the topology defined by d. Show that $\delta(x, y) \leq 1$ for all $(x, y) \in X \times X$. (This shows that in any metric space, one can change the distance so that any two points are at distance ≤ 1 without changing the topology.)

(2) Let $X = \mathbf{R}^2$. Let d denote the euclidean distance on X. Define
$$\delta(x, y) = \begin{cases} d(x, y) & \text{if } x \text{ and } y \text{ are proportional,} \\ d(x, 0) + d(0, y) & \text{if } x \text{ and } y \text{ are not proportional} \end{cases}$$
for $(x, y) \in \mathbf{R}^2$.

  (a) Show that δ is a distance on $\mathbf{R}^2$.

  (b) Give a geometric description of the sets
$$B_\delta((x_0, y_0), r) = \{(x, y) \in \mathbf{R}^2 \mid \delta((x_0, y_0), (x, y)) < r\}.$$

  (c) Show that any open set for the euclidean topology $\mathscr{T}_d$ is an open set for the topology $\mathscr{T}_\delta$ defined by δ.

  (d) Show that the converse is false: there are open sets for $\mathscr{T}_\delta$ which are not open for $\mathscr{T}_d$.

(3) Let X be a set.

  (a) Show that a topology $\mathscr{T}$ on X is the discrete topology if and only if, for all $x \in X$, the singleton $\{x\}$ is open for $\mathscr{T}$.

  (b) Find a metric d on X such that the topology defined by d is the discrete topology on X.

(4) Let C be the Cantor space of sequences $(x_n)_{n\geq 1}$ with $x_n \in \{0,1\}$, with the topology defined in the lecture.

(a) Show that the map
$$t\colon C \to [0,1]$$
such that
$$t((x_n)) = \sum_{n\geq 1} \frac{2x_n}{3^n}$$
is well-defined and that it is continuous and injective.

(b) Show that the image of t is closed in $[0,1]$. (Hint: show that the complement is open; for this, you can use that if $x \in [0,1]$ is not in the image of C, then there is a ternay expansion
$$x = \sum_{n\geq 1} \frac{a_n}{3^n}$$
where $a_n \in \{0,1,2\}$ and at least one a_n is equal to 1.)

(c) Show that the reciprocal map $t^{-1}\colon t(C) \to C$ is continuous. (So that C is homeomorphic to the subset $t(C)$ of $[0,1]$.) Remark: soon, we will see that (b) and (c) follow immediately from the abstract property that C is *compact*.