TOPOLOGY SPRING 2024
SERIE 3

(1) Let X be the space of *all* functions from $\mathbf{R}$ to $\mathbf{C}$ with the topology of pointwise convergence $\mathscr{T}_p$.

(a) Let $A \subset X$ be any subset. Show that $0 \in \bar{A}$ if and only if, for any $N \geq 1$, any $(x_i) \in \mathbf{R}^N$ and any $\varepsilon > 0$, there exists $f \in A$ such that $|f(x_i)| < \varepsilon \ \forall i$.

(b) Let $A \subset X$ be the subset of monic polynomials with real coefficients. Show that $0 \notin A$ and that $0 \in \bar{A}$. Show that the same holds for A the subset of the indicator functions of the complements of finite sets (i.e $f_S(x) = 1$ for $x \notin S$ and 0 otherwise, with S finite).

(c) For the set A of the indicator functions in the previous point, show that there exists no sequence (f_n) with $f_n \in A$ for all n such that $f_n \to 0$ as $n \to +\infty$ (for the topology $\mathscr{T}_p$).

(2) Let X be the space of *all* functions from $\mathbf{C}$ to $\mathbf{C}$, and denote by $\mathscr{T}_p$ and $\mathscr{T}_u$ the topologies of pointwise convergence and uniform convergence, respectively. Let $A \subset X$ be the subset of polynomial functions $f\colon \mathbf{C} \to \mathbf{C}$.

(a) Show that A is dense in $(X, \mathscr{T}_p)$.

(b) Show that $\mathring{A}$ is empty in $(X, \mathscr{T}_p)$.

(c) Show that for any $f \in X$, the sets

$$V_{f,n} = \{g \in X \mid \sup_{\mathbf{C}} |f(x) - g(x)| < 1/n\}$$

for $n \geq 1$ form a countable fundamental system of neighborhoods of f in $(X, \mathscr{T}_u)$.

(d) Show that the subset

$$A_0 = \{f \in A \mid f(0) = 0\}$$

is discrete in $(X, \mathscr{T}_u)$, i.e., the subspace topology on A_0 induced by the topology $\mathscr{T}_u$ is the discrete topology.

(3) Let $n \geq 0$ be an integer. A subset A of $\mathbf{C}^n$ is called *algebraic* if there exists a set I (potentially arbitrary) and a family $(f_i)_{i \in I}$ of polynomials $f_i \in \mathbf{C}[X_1, \ldots, X_n]$ such that

$$A = \{(x_1, \ldots, x_n) \in \mathbf{C}^n \mid f_i(x) = 0 \text{ for all } i \in I\}.$$

(a) Show that there is a topology $\mathscr{T}_Z$ (the "Zariski topology") on $\mathbf{C}^n$ such that $A \subset \mathbf{C}$ is closed if and only if A is algebraic.

(b) Show that for $n = 1$, the Zariski topology on $\mathbf{C}$ is identical with the topology $\mathscr{T}_{fin}$ with closed sets given by $\mathbf{C}$ and finite sets.

(c) Let $m \geq 0$ be an integer and let $f\colon \mathbf{C}^n \to \mathbf{C}^m$ be a polynomial map (i.e., $f(x) = (f_1(x), \ldots, f_m(x))$ where each f_i is a polynomial in $\mathbf{C}[X_1, \ldots, X_n]$). Show that f is continuous for the Zariski topologies.

(d) For $n \geq 1$, show that the Zariski topology on $\mathbf{C}^n$ is not Hausdorff.

(e) Show that $A \subset \mathbf{C}^n$ is dense for the Zariski topology unless there exists $f \in \mathbf{C}[X_1, \ldots, X_n]$, $f \neq 0$, such that

$$A \subset \{x \in \mathbf{C}^n \mid f(x) = 0\}.$$

(f) Show that $\mathbf{Z}^n$ is dense in $\mathbf{C}^n$ for the Zariski topology. (Hint: use the previous question, and argue by induction on n, writing a polynomial f vanishing on $\mathbf{Z}^n$ as a polynomial in X_n with coefficients in $\mathbf{C}[X_1, \ldots, X_{n-1}]$ for the induction step.)

(4) Let $n \geq 1$ be an integer. The goal of this exercise is to show that if $U \subset \mathbf{C}^n$ is any non-empty open set for the Zariski topology, then U is dense for the Zariski topology. We argue by contradiction, so assume that $\bar{U} \neq \mathbf{C}^n$.

(a) Show that there exist closed sets $A_1 \neq \mathbf{C}^n$ and $A_2 \neq \mathbf{C}^n$ such that $A_1 \cup A_2 = \mathbf{C}^n$.

(b) Let

$$I_i = \{f \in \mathbf{C}[X_1, \ldots, X_n] \mid f(x) = 0 \text{ for all } x \in A_i\}$$

for $i = 1, 2$. Show that $I_1 \cap I_2 = \{0\}$.

(c) Deduce that either $I_1 = \{0\}$ or $I_2 = \{0\}$, and derive a contradiction. (Hint: if both are non-zero, consider a product $f_1 f_2$ with $f_i \in I_i$ non-zero.)

(5) Let $n \geq 1$ be an integer. We identify the space $M_n(\mathbf{C})$ of $n \times n$ matrices with complex coefficients with the space $\mathbf{C}^{n^2}$. Show that $\mathrm{GL}_n(\mathbf{C}) \subset M_n(\mathbf{C})$ is open. Deduce that any polynomial function of the entries of a matrix which vanishes for all invertible matrices is the zero polynomial, so vanishes for all matrices. (Hint: use the previous exercise.)