TOPOLOGY SPRING 2024
SERIE 4

(1) Let X be a topological space. Show that X is Hausdorff if and only if, for every $x \in X$, we have

$$\bigcap_{x \in U} \overline{U} = \{x\}$$

where U runs over all neighborhoods of x.

(2) Let X be a topological space.

(a) If X is a Hausdorff space, show that a filter $\mathfrak{F}$ on X which converges has a unique limit.

(b) Conversely, suppose a convergent filter $\mathfrak{F}$ on X always has a unique limit. Show that X is a Hausdorff space. (Hint: consider the set

$$\mathfrak{F} = \{\emptyset \neq A \subset X \mid A \supset V \cap W \text{ for some open neighborhood } V \text{ of } x \text{ and } W \text{ of } y\},$$

and show that it is a filter on X.)

(3) Let X and Y be topological spaces. Define a topology on $X \times Y$ by saying that $U \subset X \times Y$ is open if and only if, for every $(x, y) \in U$, there exist neighborhoods V and W of x and y respectively such that $V \times W \subset U$ (this is the product topology).

(a) Check that this is a topology on $X \times Y$.

(b) Show that X is a Hausdorff space if and only if the diagonal

$$\Delta = \{(x, x) \mid x \in X\} \subset X \times X$$

is closed in $X \times X$ (with the above topology for $Y = X$).

In the following points, let Y be a Hausdorff space.

(c) For any topological space X and any continuous maps $f\colon X \to Y$ and $g\colon X \to Y$, show that the sets

$$\{(x, y) \mid f(x) = g(y)\}, \qquad \{x \in X \mid f(x) = g(x)\}$$

are closed in $X \times X$ and in X, respectively.

(d) For any topological space X and any continuous maps $f\colon X \to Y$ and $g\colon X \to Y$, show that if $f(x) = g(x)$ for all x in a dense set, then $f = g$.

(e) For any topological space X and any continuous map $f\colon X \to Y$, show that the graph

$$\Gamma_f = \{(x, y) \mid y = f(x)\}$$

of f is closed in $X \times Y$.

(4) Let $X = \mathbf{R}$ with the cofinite topology.

(a) Show that for every $x \neq y$ in X, there exists a neighborhood U of x such that $y \notin U$, but that X is not Hausdorff.

(b) Show that the graph of the identity map $X \to X$ is not closed in $X \times X$ (with the product topology).

(c) Show that a function $f\colon X \to X$ is continuous if *either* f is constant, *or* if the equation $f(x) = y$ has at most finitely many solutions for arbitrary $y \in \mathbf{R}$. In particular, any bijection $X \to X$ is continuous.

(d) Find examples of two bijections $f\colon X \to X$ and $g\colon X \to X$ such that $f - g$ is *not* continuous.

(e) Deduce examples showing that the properties of (c) and (d) of the previous exercise are not always true if X is not Hausdorff.

(5) Let X be a topological space.

(a) If A_1 and A_2 are compact subsets of X, show that $A_1 \cup A_2$ is compact.

(b) If X is Hausdorff, show that $A_1 \cap A_2$ is compact.

(6) Let X and Y be topological spaces, with Y compact and Hausdorff. Let $f\colon X \to Y$ be an arbitrary map such that the graph Γ_f of f is *closed* in $X \times Y$ (with the topology of Exercise 3.)

(a) Let $x_0 \in X$ and denote $y_0 = f(x_0)$. For any $y \neq y_0$ in Y, show that there exist open neighborhoods U_y of x_0 and V_y of y such that $(U_y \times V_y) \cap \Gamma_f = \emptyset$. (Hint: note that $(x_0, y) \notin \Gamma_f$; use the assumption and the definition of the product topology.)

(b) Let V be an open neighborhood of y_0 in Y. Deduce from (a) that there exists an open neighborhood U of x_0 such that

$$(U \times (Y \setminus V)) \cap \Gamma_f = \emptyset.$$

(Hint: use compactness of Y.)

(c) Prove that f is continuous. (This is an example of a *closed graph theorem*, a statement which shows that certain maps are continuous as soon as their graphs are closed.)

(7) Let X be a topological space. Show that X is compact if and only if, for any family $(U_x)_{x \in X}$ of open sets such that U_x is an open neighborhood of x for all x, there exists a finite subset $S \subset X$ such that

$$X = \bigcup_{x \in S} U_x.$$

(In other words, it is sufficient to consider the type of open coverings seen in many examples, indexed by X, and with U_x a neighborhood of x.)