TOPOLOGY SPRING 2024
SERIE 5

(1) Let X be a topological space.

(a) If X is Hausdorff, show that $\{x\}$ is closed in X for all $x \in X$.

We define a set $\widetilde{X} = X \cup \{\eta\}$, where η is any mathematical object not in X. We define a topology on $\widetilde{X}$ so that U is open if and only if either $U = \emptyset$, or $U = V \cup \{\eta\}$ for some open set $V \subset X$.

(b) Check that this defines a topology, and that the inclusion map $X \to \widetilde{X}$ is continuous.

(c) Show that the closure of $\{\eta\}$ is equal to $\widetilde{X}$, i.e., that the single point η is dense in $\widetilde{X}$.

(2) Let X be a compact Hausdorff space.

(a) Let $x \in X$ and $A \subset X$ be a closed subspace with $x \notin A$. Show that there exists a neighborhood U of x such that $\overline{U} \cap A = \emptyset$. (Hint: adapt the proof that a compact subset of X is closed to find an open neighborhood U of x and an open set V containing A such that $U \cap V = \emptyset$.)

Let $(C_n)_{n \geq 1}$ be a sequence of closed subsets of X with empty interior for all n. Denote

$$C = \bigcup_{n \geq 1} C_n.$$

Let U be a non-empty open subset of X.

(b) Let $U_0 = U$. Show that one can construct by induction a sequence of open set $(U_n)_{n \geq 1}$ such that, for all $n \geq 1$, the properties

$$\begin{cases} \overline{U}_n \cap C_n = \emptyset \\ \overline{U}_n \subset U_{n-1} \end{cases}$$

are satisfied. (Hint: use the fact that each C_n has empty interior, so does not contain any non-empty open set, and (a).)

(c) Show that

$$\bigcap_{n \geq 1} \overline{U}_n \neq \emptyset,$$

and deduce that $U \cap (X \setminus C) \neq \emptyset$.

(d) Deduce that the interior of C is empty (this is known as the *Baire property*).

(e) Show that if (V_n) is a sequence of dense open sets in X, the intersection

$$\bigcap_{n \geq 1} V_n$$

is still dense in X.

(f) Give an example of a sequence (V_n) of dense open sets in the compact space $[0, 1]$ such that the intersection of the V_n's is not open.

(3) Let $X = \mathbf{R} \times \{-1, 1\}$ with the product topology (where $\{-1, 1\}$ has the discrete topology). We define an equivalence relation on X so that

$$(x, 1) \sim (x, -1) \text{ if } x \neq 0,$$

and there are no further equivalences except equality. (In particular, the equivalence classes o_+ and o_- of the points $(0, 1)$ and $(0, -1)$ have only one element, and give different points in Y.)

Let $Y = X/\sim$ be the space of equivalence classes. Let $p\colon X \to Y$ be the quotient map; define a topology $\mathscr{T}$ on Y so that $U \subset Y$ is open if and only if $p^{-1}(U) \subset X$ is open.

(a) Show that this defines a topology on Y.
(b) For $\varepsilon \in \{-1, 1\}$, define a map $i_\varepsilon\colon \mathbf{R} \to Y$ that sends x to the equivalence class of (x, ε). Show that i_ε is continuous and injective.
(c) Show that i_+ has image $Y \setminus \{o_-\}$ and gives a homeomorphism $\mathbf{R} \to Y \setminus \{o_-\}$. Similarly, i_- defines a homeomorphism $\mathbf{R} \to Y \setminus \{o_+\}$.
(d) Show that Y is a topological manifold of dimension 1 (i.e., for every $y \in Y$, there exists an open neighborhood of y which is homeomorphic to an open subset of $\mathbf{R}$).
(e) Show that every $y \in Y$ has a countable fundamental system of neighborhoods.
(f) Show that Y is *not* Hausdorff. In particular, find a sequence (y_n) in Y which converges to both o_+ and o_-.

(4) Let X be a compact topological space. We denote by $\mathscr{C}(X)$ the set of continuous functions $f\colon X \to \mathbf{C}$, where $\mathbf{C}$ has the euclidean topology.

(a) Show that $\mathscr{C}(X)$ is a commutative ring, with addition given by $(f + g)(x) = f(x) + g(x)$, multiplication by $(fg)(x) = f(x)g(x)$ and neutral element for multiplication the constant function 1.

Let $I \subset \mathscr{C}(X)$ be an ideal (i.e., a subgroup for addition such that $fg \in I$ whenever $f \in I$ and $g \in \mathscr{C}(X)$). *The goal of the exercise is to show that either $I = \mathscr{C}(X)$ or there exists some $x_0 \in X$ such that*

$$I \subset m_{x_0} = \{f \in \mathscr{C}(X) \mid f(x_0) = 0\}.$$

(b) Show that the function $x \mapsto |f(x)|^2$ is in I for all $f \in I$.
(c) Show that m_{x_0} is an ideal whenever $x_0 \in X$.
(d) Show that if there exists a function $f \in I$ such that $f(x) \neq 0$ for all $x \in X$, then $I = \mathscr{C}(X)$.
(e) Suppose there is no x_0 such that $I \subset m_{x_0}$. Deduce that for every $x \in X$, there exists an open neighborhood U_x of x and a function $f_x \in I$ such that $f_x(y) \neq 0$ for all $y \in U_x$.
(f) Deduce that if there is no x_0 such that $I \subset m_{x_0}$, then $I = \mathscr{C}(X)$. (Hint: construct, using the compactness property and the f_x's, a function in I which has no zero in X.)
(g) Show that I is a maximal ideal (i.e., an ideal different from $\mathscr{C}(X)$ which is not properly contained in any other ideal) if and only if $I = m_{x_0}$ for some $x_0 \in X$.

(The results of this exercise are related to the *Gelfand equivalence* between compact topological spaces and certains abstract rings.)