TOPOLOGY SPRING 2024
SERIE 6

(1) Let X be a non-empty set.
   (a) Show that an ultrafilter $\mathscr{F}$ on X is principal if and only if there exists a finite set $A \subset X$ such that $A \in \mathscr{F}$.
   (b) Show that if X is infinite, then there exists a non-principal ultrafilter on X. (Hint: consider the complements of finite sets.)
   (c) Let $\mathscr{F}$ be an ultrafiler on X. Show that if A, B are subsets of X, then $A \cup B \in \mathscr{F}$ if and only if $A \in \mathscr{F}$ or $B \in \mathscr{F}$. (Hint: if $A \cup B \in \mathscr{F}$ and $A \notin \mathscr{F}$, show that $B \setminus A \in \mathscr{F}$.)
   (d) For $\mathscr{F}$ as above, define a function μ from the subsets of X to $\{0, 1\}$ by

$$\mu(A) = \begin{cases} 1 & \text{if } A \in \mathscr{F} \\ 0 & \text{otherwise.} \end{cases}$$

   Show that for any subsets A, B of X with $A \cap B = \emptyset$, we have

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

   (e) Conversely, let ν be a function from the subsets of X to $\{0, 1\}$ such that $\nu(X) = 1$ and

$$\nu(A \cup B) = \nu(A) + \nu(B)$$

   whenever A, B are disjoint subsets of X.
   Show that

$$\nu(A \cap B) + \nu(A \cup B) = \nu(A) + \nu(B)$$

   for any subsets A and B of X.
   (f) Show that

$$\mathscr{F}_\nu = \{A \subset X \mid \nu(A) = 1\}$$

   is an ultrafilter on X.
   (These last questions show that ultrafilters on X can be identified with some kinds of finitely-additive measures on X.)

(2) (a) Let X be a topological space and let A be a connected subset of X. If B is a subset of X such that $A \subset B \subset \overline{A}$, show that B is connected.
   (b) Let

$$A = \{(x, y) \in \mathbf{R}^2 \mid x > 0 \text{ and } y = \sin(1/x)\} \subset \mathbf{R}^2.$$

   Show that A is connected.
   (c) Let

$$B = A \cup \{(0, y) \mid -1 \leq y \leq 1\} \subset \mathbf{R}^2.$$

   Show that B is connected.

(3) Let X be a topological space and A a subset of X.

(a) Show that

$$X - \partial A = A^\circ \cup (X \setminus A)^\circ$$

where $\partial A = \overline{A} \cap \overline{X \setminus A}$ is the boundary of A.

(b) Let $B \subset X$ be a *connected* subspace. If $B \cap \partial A$ is empty, show that $B \subset A^\circ$ or $B \subset (X \setminus A)^\circ$.

(c) For B as above, if $B \cap A$ and $B \cap (X \setminus A)$ are both non-empty, deduce that $B \cap \partial A \neq \emptyset$.

(d) If X is connected, and A is not empty nor equal to X, show that ∂A is not empty.

(e) If X is not connected, find a subset A, not empty nor equal to X, with empty boundary.

(4) Let C be the Cantor space. Show that if $A \subset C$ is connected, then A is either empty or a single point. (Hint: consider the projections $p_n \colon C \to \{0, 1\}$ mapping an element of C to its n-th entry.)

(5) Let $d \geq 0$ be an integer.

(a) Show that $\mathbf{R}^d$ is connected. (Hint: consider the cases $d = 0$ and $d = 1$ first; then for $d \geq 2$, consider a continuous function $f \colon \mathbf{R}^d \to \{0, 1\}$ and show it is constant by composing with suitable maps $\mathbf{R} \to \mathbf{R}^d$.)

(b) If $d = 1$, show that $\mathbf{R}^d \setminus \{0\}$ is not connected. What are its connected components?

(c) If $d \geq 2$, show that $\mathbf{R}^d \setminus \{0\}$ is connected. (Hint: use ideas like those in (a)).

(d) Let $\mathbf{S}_d \subset \mathbf{R}^{d+1}$ be the set of $(x_1, \ldots, x_{d+1}) \in \mathbf{R}^{d+1}$ such that

$$x_1^2 + \cdots + x_{d+1}^2 = 1.$$

Show that $\mathbf{S}_d$ is connected if and only if $d \geq 1$. (Hint: find a surjective continuous map $\mathbf{R}^{d+1} \setminus \{0\} \to \mathbf{S}_d$.)

(e) Let $r \geq 0$. Show that the closed ball

$$B_r = \{x \in \mathbf{R}^d \mid 0 \leq \sqrt{x_1^2 + \cdots + x_d^2} \leq r\}$$

is connected. (Hint: one option is to find connected subsets C_s for $0 \leq s \leq r$ such that

$$B_r = \bigcup_{0 \leq s \leq r} C_s$$

and such that the C_s have some common points.)

(6) Let

$$X = \mathbf{S}_1 \setminus \{(0, 1)\} \subset \mathbf{R}^2.$$

(a) Show that X, with the subspace topology, is a connected metric space.

(b) Show that there exist some balls in X which are not connected.

(7) This exercise gives a different presentation of the proof that $[a, b]$ is compact in $\mathbf{R}$, using the connectedness of intervals. Let $(U_i)_{i \in I}$ be open subsets in $\mathbf{R}$ such that

$$[a, b] \subset \bigcup_{i \in I} U_i.$$

(a) Show that the set

$$G = \{t \in [a, b] \mid [a, t] \text{ is contained in } \bigcup_{j \in J} U_j \text{ some some finite } J \subset I\}$$

is open in $[a, b]$.

(b) Show that G is closed in $[a, b]$.

(c) Conclude that $G = [a, b]$, and hence that $[a, b]$ is compact.