TOPOLOGY SPRING 2024
SERIE 7

(1) Let $X = [0, +\infty[$. Define a function

$$\delta \colon X \times X \to [0, +\infty[$$

by

$$\delta(x, y) = |e^{-x} - e^{-y}|.$$

(a) Show that δ is a distance on X.
(b) Show that the topology defined by δ on X is the euclidean subspace topology. (Hint: consider the map $f \colon (X, \delta) \to]0, 1]$ defined by $f(x) = e^{-x}$ and show that it is a homeomorphism.)
(c) Show that (X, δ) is *not* complete.
(d) Show that X is a complete space with the restriction of the euclidean distance.

This exercise illustrates the fact the *completeness* of a metric space does not simply depend on the topology.

(2) In a metric space (X, d), the *diameter* of a non-empty subset $Y \subset X$ is defined to be

$$\operatorname{diam}(Y) = \sup\{d(x, y) \mid (x, y) \in Y^2\}$$

if this supremum exists in $\mathbf{R}$, or $+\infty$ otherwise.

Let X be a complete metric space. Let $(C_n)_{n \geq 1}$ be closed subsets of X such that $C_n \supset C_{n+1}$ for $n \geq 1$.

(a) Assuming that $C_n \neq \emptyset$ and that

$$\lim_{n \to +\infty} \operatorname{diam}(C_n) = 0,$$

show that

$$\bigcap_{n \geq 1} C_n \neq \emptyset.$$

(Hint: show that if (x_n) is a sequence with $x_n \in C_n$, then it is a Cauchy sequence, and that its limit is in the intersection.)
(b) Show that the conclusion is not always valid if X is not complete.

(3) Let X be a complete metric space. Let $(C_n)_{n \geq 1}$ be a sequence of closed subsets of X, each with empty interior. Let C be the union of the C_n for $n \geq 1$.

Let U be a non-empty open subset of X.

(a) Let $U_0 = U$. Show that there exists a sequence $(U_n)_{n \geq 1}$ of non empty open sets in X such that the conditions

$$\begin{cases} \overline{U}_n \cap C_n = \emptyset \\ \overline{U}_n \subset U_{n-1} \\ d(x, y) < \frac{1}{n} \text{ for all } (x, y) \in U_n \times U_n \end{cases}$$

hold for $n \geq 1$.

(b) Show that
$$\bigcap_{n\geq 1} \overline{U}_n \neq \emptyset.$$
(c) Deduce that $U \cap (X \setminus C) \neq \emptyset$, and therefore that C has empty interior. (This is *Baire's Theorem* for complete metric spaces; a version for compact spaces was in Exercise 2 of Exercise sheet 5.)

(4) Let X be a non-empty complete metric space and let $f\colon X \to X$ be a continuous map such that
$$d(f(x), f(y)) \leq \alpha d(x, y)$$
for all $(x, y) \in X^2$ for some constant $\alpha < 1$.
(a) Show that for any x_0 in X, the sequence (x_n) defined by $x_{n+1} = f(x_n)$ converges to a solution $y \in X$ of the equation $f(y) = y$.
(b) Show that there is a unique element $y \in X$ such that $f(y) = y$.

(5) Let
$$X = (\{0\} \times [-1, 1]) \;\cup\; \bigcup_{n\geq 1} \{1/n\} \times [-1, 1] \;\cup\; ([0, 1] \times \{1\}) \subset \mathbf{R}^2$$
with the subspace topology from the euclidean topology of $\mathbf{R}^2$. It is a metric space.
(a) Show that X is connected. (Hint: it is useful to sketch the set X in $\mathbf{R}^2$.)
(b) Let $x_0 = (0, 0) \in X$. Explain why the sets
$$U_\delta = X \cap \{(x, y) \in \mathbf{R}^2 \mid |x| < \delta, \quad |y| < \delta\}$$
with $0 < \delta < 1$ form a fundamental system of open neighborhoods of x_0 in X.
(c) Show that if $0 < \delta < 1$, then the connected component of x_0 in the neighborhood U_δ is $\{0\} \times]{-\delta}, \delta[$.
(d) Deduce that X is *not* locally connected.
(e) Give an example of a locally connected space which is not connected.

(6) Recall that any subspace of a compact Hausdorff space is locally compact. This exercise will show a converse: any locally compact Hausdorff space is homeomorphic to a subspace of a compact Hausdorff topological space.

Let X be a locally compact Hausdorff space. Let ∞ denote any mathematical object not in X. Define $\widehat{X} = X \cup \{\infty\}$. Let $i\colon X \to \widehat{X}$ be the inclusion map.
(a) Let $\mathscr{T}_\infty$ be the set of subsets U of $\widehat{X}$ which are either open subsets of X, or sets of the form $\{\infty\} \cup (X \setminus C)$ for $C \subset X$ compact. Show that $\mathscr{T}_\infty$ is a topology on $\widehat{X}$. (Exercise 5 of Exercise Sheet 4 will be useful, but will need to be partly generalized.)
(b) Show that $\widehat{X}$ is a Hausdorff space. (Hint: to find neighborhoods of $x \neq y$ which are disjoint, consider the case where x or y is ∞ separately.)
(c) Show that i is continuous and that i defines a homeomomorphism $X \to i(X) \subset \widehat{X}$.
(d) Show that $\widehat{X}$ is compact.

(e) Where in this proof was it used that X is locally compact? What happens if X is assumed to be compact?

The space $\widehat{X}$ is called the *Alexandrov compactification* of X.