TOPOLOGY SPRING 2024
SERIE 8

(1) Let $X = L^2([0,1])$ with the topology defined by the distance

$$d(f,g) = \Big(\int_0^1 |f-g|^2 dx\Big)^{1/2}.$$

(a) For $f \in X$ and $\varepsilon > 0$, show that the closed ball

$$B(f;\varepsilon) = \{g \in X \mid d(f,g) \leq \varepsilon\}$$

is *not* compact. (Hint: start with $f = 0$ and $\varepsilon = 1$, which was an example in the lecture.)

(b) Deduce that X is *not* locally compact, and in fact that there is no $f \in X$ which has any compact neighborhood.

(2) Let $(X_i)_{i\in I}$ be a family of topological spaces and let

$$X = \prod_{i\in I} X_i,$$

with the product topology.

(a) If X_i is Hausdorff for all i, prove that X is also Hausdorff.

(b) Let $Y_i \subset X_i$ be an arbitrary subset for each i. Show that the subspace topology on

$$Y = \prod_{i\in I} Y_i \subset X$$

is the product of the subspace topologies of Y_i.

(c) Let $Y_i \subset X_i$ be an arbitrary subset for each i. Show that

$$\overline{\prod_{i\in I} Y_i} = \prod_{i\in I} \overline{Y}_i.$$

(d) If $C_i \subset X_i$ is closed for all i show that the subset

$$\prod_{i\in I} C_i$$

is closed in X.

(e) Give an example of a set I, spaces X_i and open subsets $U_i \subset X_i$ such that

$$\prod_{i\in I} U_i$$

is not open in X.

(f) Let $x_n = (x_{n,i})_{i\in I}$ be elements of X for all $n \geq 1$. Show that the sequence (x_n) converges to an element $x = (x_i)_{i\in I}$ of X if and only if $x_{n,i} \to x_i$ as $n \to \infty$ for all $i \in I$.

(g) For any $x = (x_i)$ in X, show that the connected component of X is equal to the product of the connected components Y_i of x_i in X_i.

(3) Let $(X_n, d_n)_{n\geq 1}$ be a sequence of metric spaces. Denote

$$X = \prod_{n\geq 1} X_n.$$

(a) Show that for $x = (x_n)$ and $y = (y_n)$ in X, the series

$$d(x,y) = \sum_{n\geq 1} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}$$

is (absolutely) convergent and that the function $d \colon X \times X \to [0, +\infty[$ it defines is a distance on X.

(b) Show that the topology defined by d is the product topology on X.

(c) Show that if X_n is complete for all n, then X is complete. (This fact is also true for an arbitrary product of complete spaces in the sense of uniform structures.)

(d) Assume that X_n is compact for all n. Show that if $x_m = (x_{m,n})_{n\geq 1}$ is an element of X for all $m \geq 1$, then the sequence $(x_m)_{m\geq 1}$ has a convergent subsequence. (Hint: show that for every $N \geq 1$, there exists a sequence $x^{(N)} = (x_k^{(N)})_{k\geq 1}$ of elements of X such that (1) $x^{(1)} = (x_m)$; (2) $x^{(N)}$ is a subsequence of $x^{(N-1)}$; (3) for $1 \leq n \leq N$, the sequence of n-th coordinates

$$(x_k^{(N)})_n$$

converges as $k \to +\infty$. To conclude, construct a convergence subsequence of (x_m) by a diagonal argument.)

(e) Deduce that X is compact without using Tychonov's Theorem.

(4) Let X_1 and X_2 be topological spaces and $X = X_1 \times X_2$ with the product topology.

(a) Let Y be a topological space and $f \colon X \to Y$ a continuous map. For any $(x_1, x_2) \in X_1 \times X_2$, show that the maps

$$f_{x_2} \colon \begin{cases} X_1 \to Y \\ x \mapsto f(x, x_2) \end{cases}, \qquad g_{x_1} \colon \begin{cases} X_2 \to Y \\ x \mapsto f(x_1, x) \end{cases}$$

are continuous.

(b) Let $X_1 = X_2 = \mathbf{R}$ and define $f \colon X \to \mathbf{R}$ by

$$f(x_1, x_2) = \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2}, & \text{if } (x_1, x_2) \neq (0,0) \\ 0 & \text{if } (x_1, x_2) = (0,0). \end{cases}$$

Show that f_{x_2} and g_{x_1} are all continuous but that f is not continuous.

Let $X_1 = X_2 = Y = \mathbf{R}$. Assume that the functions f_{x_2} and g_{x_1} are continuous for all $(x_1, x_2) \in \mathbf{R}^2$. Let $(x_1, x_2) \in \mathbf{R}^2$ and $y = f(x_1, x_2)$.

(c) For $\varepsilon > 0$, show that there exist $y_1 < y_2$ in $\mathbf{R}$ with $y_1 < x_2 < y_2$ such that $y - \varepsilon < f(x_1, x) < y + \varepsilon$ if $y_1 < x < y_2$.

(d) Let $v_1 < v_2$ be such that $y_1 < v_1 < x_2 < v_2 < y_2$. Show that there exists $\delta > 0$ such that

$$y - \varepsilon < f(x, v_1) < y + \varepsilon$$
$$y - \varepsilon < f(x, v_2) < y + \varepsilon$$

for $x_1 - \delta < x < x_1 + \delta$.

(e) Assume furthermore that $g_{x_1} : \mathbf{R} \to \mathbf{R}$ is non-decreasing for all $x_1 \in \mathbf{R}$. Deduce that f is continuous.