TOPOLOGY SPRING 2024
SERIE 9

In these exercises, we use the following notion: a map $f\colon X \to Y$ between two topological spaces is called *open* if $f(U)$ is open in Y for all open subsets U of X.

(1) For $z \in \mathbf{C}$ and $r \geq 0$, we denote by $C(z,r)$ the circle centered at z of radius r in $\mathbf{C}$, i.e. $C(z,r) = \{w \in \mathbf{C} \mid |w - z| = r\}$.

Let $X = [0,2] \subset \mathbf{R}$ and $Y = C(i,1) \cup C(-i,1)$.

(a) Show that the map $\varphi\colon X \to Y$ such that

$$\varphi(t) = \begin{cases} i + e^{2i\pi(t-1/4)} & \text{if } 0 \leq t \leq 1 \\ -i + e^{2i\pi(t-3/4)} & \text{if } 1 \leq t \leq 2 \end{cases}$$

is well-defined, and that it is continuous.

(b) Define an equivalence relation on X with equivalence classes $\{0,1,2\}$ and $\{t\}$ for $t \in X \setminus \{0,1,2\}$. Let $X' = X/\sim$ and $p\colon X \to X'$ the projection. Show that there is a continuous map $\widetilde{\varphi}\colon X' \to Y$ such that $\widetilde{\varphi} \circ p = \varphi$, and that $\widetilde{\varphi}$ is continuous when X' has the quotient topology.

(c) Show that $\widetilde{\varphi}$ is a homeomorphism.

(2) Let X and Y be topological spaces with equivalence relations $\sim_X$ and $\sim_Y$ on X and Y respectively. Let $Z = X \times Y$ with the product topology. On Z, let $\sim$ be the equivalence relation defined by

$$(x_1, y_1) \sim (x_2, y_2) \quad \text{if and only if} \quad (x_1 \sim_X x_2 \text{ and } y_1 \sim_Y y_2).$$

Let

$$X' = X/\sim_X, \quad Y' = Y/\sim_Y, \quad Z' = Z/\sim$$

denote the respective quotients. Each is given the quotient topology. Let finally p_X, p_Y and p denote the projections

$$p_X\colon X \to X', \quad p_Y\colon Y \to Y', \quad p\colon Z \to Z'.$$

(a) Show that there is a bijection

$$\varphi\colon Z' \to X' \times Y'$$

such that the class of (x,y) is mapped to $(p_X(x), p_Y(y))$, and that φ is continuous when $X' \times Y'$ has the product topology.

(b) Suppose that p_X and p_Y are *open* maps. Show that $\varphi(p(W))$ is open in $X' \times Y'$ for all $W \subset Z$ open.

(c) Deduce that φ is an homeomorphism in that case.

(3) Let X be a topological space and $\sim$ an equivalence relation on X. Let $X' = X/\sim$ and $p\colon X \to X'$ the projection. Let

$$\Gamma = \{(x,y) \in X \times X \mid x \sim y\}$$

be the *graph* of the equivalence relation. Define the relation $\equiv$ on $X \times X$ by

$$(x, y) \equiv (x', y') \quad \text{if and only if} \quad x \sim x' \text{ and } y \sim y'.$$

(a) Show that $\Gamma = q^{-1}(q(\Delta))$ where

$$\Delta = \{(x, x) \in X \times X \mid x \in X\}$$

is the diagonal in $X \times X$ and $q\colon X \times X \to (X \times X)/\equiv$ is the projection.

(b) If the map p is open and Γ is closed in $X \times X$, show that the diagonal

$$\Delta' = \{(x, x) \in X' \times X' \mid x \in X'\}$$

is closed in $X' \times X'$ for the product topology. (Hint: use the previous exercise.)

(c) Deduce that the space X' is then Hausdorff. (Hint: use Exercise 3 of Exercise sheet 4).

(4) We use the setting and notation of Exercise 2, with $X = Y = \mathbf{R}$ (with the euclidean topology), and let $\sim_X$ be the equality relation, $\sim_Y$ the relation where the equivalence class of 0 is $\mathbf{Z}$ while the equivalence class of any $x \notin \mathbf{Z}$ is $\{x\}$.

(a) Show that p_X is open.

(b) Show that if $A \subset Y$ is any subset, then

$$p_Y^{-1}(p_Y(A)) = \begin{cases} A & \text{if } A \cap \mathbf{Z} = \emptyset \\ \mathbf{Z} \cup A & \text{otherwise.} \end{cases}$$

(c) Deduce that p_Y is not an open map. Show however that $p_Y(C)$ is closed if C is closed.

(d) Show that a fundamental system of neighborhoods of $(p_X(0), p_Y(0))$ in $X' \times Y'$ is given by the sets of the form

$$p_X(]-\delta, \delta[) \times p_Y\Bigl(\bigcup_{n \in \mathbf{Z}}]n - \varepsilon_n, n + \varepsilon_n[\Bigr)$$

where $\delta > 0$ and $\varepsilon_n > 0$, for $n \in \mathbf{Z}$, are arbitrary positive real numbers.

(e) Show that a fundamental system of neighborhoods of $p(0, 0)$ in Z' is given by the sets of the form

$$p\Bigl(\bigcup_{n \in \mathbf{Z}}]-\delta_n, \delta_n[\times]n - \varepsilon_n, n + \varepsilon_n[\Bigr)$$

where $\delta_n > 0$ and $\varepsilon_n > 0$, for $n \in \mathbf{Z}$, are arbitrary positive real numbers.

(f) Deduce $\varphi\colon Z' \to X' \times Y'$ is *not* a homeomorphism.

(g) Can you get some intuitive feeling for the "shape" of Y'? for that of Z'?

(5) Let $n \geq 1$ and $k \leq n$ be non-negative integers. Let $H_k \subset \mathbf{R}^n$ be the subgroup, isomorphic to $\mathbf{Z}^k$, generated by the first k vectors of the canonical basis of $\mathbf{R}^n$. Let $X_{n,k} = \mathbf{R}^n/H_k$, with the corresponding quotient topology (where H_k has the subspace topology, which is discrete). Let $p\colon \mathbf{R}^n \to X_{n,k}$ be the canonical projection.

(a) Show that p is open and that the graph of the equivalence relation is closed. Deduce that $X_{n,k}$ is a Hausdorff space. (Hint: use the criterion of Exercise 3.)

(b) Let $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$. Show that

$$C = \{(t_1, \ldots, t_n) \in \mathbf{R}^n \mid |t_i - x_i| \leq \frac{1}{4} \text{ for } 1 \leq i \leq n\}$$

is a compact neighborhood of x such that the restriction of p to C is injective.

(c) Deduce that $X_{n,k}$ is a topological manifold.

(d) Construct an homeomorphism $X_{n,k} \to (\mathbf{S}_1)^k \times \mathbf{R}^{n-k}$. (Hint: Exercise 2 can be useful.)