TOPOLOGY SPRING 2024
SOLUTIONS SERIE 1

(1) (a) Let us verify the definition of topology:
- the empty set belongs to $\mathscr{T}_{\text{cof}}$ by definition, and X belongs to it as $X \setminus X = \emptyset$ is finite;
- let $\{U_i\}_{i\in I}$ be a collection of sets in $\mathscr{T}_{\text{cof}}$, and let $U = \bigcup_{i\in I} U_i$ be their union. Let WLOG $U \neq \emptyset$ as we already know it belongs to $\mathscr{T}_{\text{cof}}$. Then I is nonempty and there is $i \in I$ such that $U_i \neq \emptyset$, but then $X\setminus U \subset X\setminus U_i$ and $X \setminus U_i$ is finite, so $X \setminus U$ is finite and hence $U \in \mathscr{T}_{\text{cof}}$;
- let $\{U_i\}_{i\in I}$ be a collection of sets in $\mathscr{T}_{\text{cof}}$ with I finite and nonempty. If one of the U_i is empty we are done as $V := \bigcap_{i\in I} U_i = \emptyset$. Otherwise, $X \setminus V = \bigcup_{i\in I}(X \setminus U_i)$ is a finite union of finite sets, and therefore a finite set, so $V \in \mathscr{T}_{\text{cof}}$.

(b) This is true precisely when X is finite: indeed, since X itself belongs to the topology, it must be a finite set. Conversely, for finite X, the set of its finite subsets is just its power set, that is, the discrete topology on X.

(2) (a) $\emptyset$ and X are respectively the preimage of $\emptyset$ and Y and therefore belong to $\mathscr{T}$. Moreover, since for any arbitrary collection $\{V_i\}_{i\in I}$ of subsets of Y (and any $f : X \longrightarrow Y$) the following equalities hold:

$$f^{-1}\left(\bigcup_{i\in I} V_i\right) = \bigcup_{i\in I} f^{-1}(V_i), \qquad f^{-1}\left(\bigcap_{i\in I} V_i\right) = \bigcap_{i\in I} f^{-1}(V_i)$$

it immediately follows that $\mathscr{T}$ satisfies the remaining two axioms of a topology: for instance, if $\{U_i\}_{i\in I}$ is a collection of elements of $\mathscr{T}$, then there exist a collection $\{V_i\}_{i\in I}$ of open subsets of Y such that $U_i = f^{-1}(V_i)$; hence, $\bigcup_{i\in I} U_i = \bigcup_{i\in I} f^{-1}(V_i) = f^{-1}(\bigcup_{i\in I} V_i) =: f^{-1}(V)$ is indeed the preimage of an open subset V of Y. The proof for the intersection property is identical, using the second equality.

(b) This is immediate from the definition of $\mathscr{T}$: if $V \subset Y$ is open, then $f^{-1}(V) \in \mathscr{T}$, so f is continuous. Indeed, it is clear that *any* topology that makes f continuous must contain $\mathscr{T}$, and we say that $\mathscr{T}$ is the *coarsest* topology that makes f continuous.

(3) (a) If U is open then $U \cap U_i$ is open for all i as it is the intersection of two open sets; moreover, since the U_is cover X we have $U = \bigcup_{i\in I}(U \cap U_i)$, so if each of the intersections is open then so is U as the union of open sets;

(b) let $f_i := f|_{U_i}$ and let $V \subset Y$ be open. Since f is continuous, then $f_i^{-1}(V) = f^{-1}(V) \cap U_i$ is the intersection of an open set of X with U_i and therefore an open set of U_i with the subspace topology, so f_i is continuous;

(c) given V as above, let $U = f^{-1}(V)$. The fact that the f_is are continuous tells us that $U_i \cap U = f_i^{-1}(V)$ is open in the subspace topology of U_i, i.e. there exists $V_i \subset X$ open such that $U_i \cap U = U_i \cap V_i$, but this last set is open in X as the intersection of two open sets, so $U_i \cap U$ is open for all i and we conclude that U itself is open by point a), so f is continuous;

(d) let $X = Y = \{0, 1\}$ with the discrete topology, $I = \{0, 1\}$, $U_i = \{i\}$, and let f be the identity. Then clearly $\{U_0, U_1\}$ is an open cover of X and f is constant on each U_i, but f is not constant;

(e) given $x \in X$, the last hypothesis allows us to unambiguously define $f(x) := f_i(x)$ for *any* $i \in I$ such that $x \in U_i$ (and such an i must exist by the covering condition), and moreover this is the only function that coincides with f_i on U_i for all i (two such function must differ on some $y \in X$, but then there exists $j \in I$ such that $y \in U_j$, so one of the two functions differs with f_j at y). Therefore, we just need to prove that such f is continuous: by construction, for all i we have $f|_{U_i} = f_i$, and the f_is are continuous by hypothesis, so we are done thanks to point c).

(4) (a) As $x \neq y$, we have $d := d(x, y) > 0$. Let $U := \{z \in X : d(x, z) < \frac{d}{2}\}$ and $V := \{z \in X : d(y, z) < \frac{d}{2}\}$, i.e. the so-called open balls of radius $\frac{d}{2}$ centred at x and y respectively. These are indeed open: fix $z \in U$ (WLOG); then $d(x, z) < \frac{d}{2} \implies \exists \epsilon > 0 : d(x, z) \leq \frac{d}{2} - \epsilon$ and hence all points $w \in X$ with $d(w, z) < \epsilon$ are in U, as for such w we have $d(x, w) \leq d(x, z) + d(z, w) < (\frac{d}{2} - \epsilon) + \epsilon = \frac{d}{2}$. The triangle inequality also shows that $U \cap V = \emptyset$: if we had $z \in U \cap V$, then $d = d(x, y) \leq d(x, z) + d(z, y) < \frac{d}{2} + \frac{d}{2} = d$, which is absurd;

(b) it clearly is enough to show that any two nonempty open sets U, V intersect each other: if that was not the case we would have $U \subset X \setminus V \implies U$ is finite as it is a subset of a finite set. But then, since $X \setminus U$ is also finite, $X = U \cup (X \setminus U)$ would also be finite as the union of two finite sets, contradicting the hypotheses.

(5)
- The first set is open, as it its open also in $\mathbf{R}$. Its complement $A = \{-1, 1\} \cup [-\frac{1}{2}, \frac{1}{2}] \subset X$ is not open, as any open set $U \subset \mathbf{R}$ that contains 1 also contains $1 - \epsilon$ for some $0 < \epsilon < \frac{1}{2}$, so its intersection with X cannot be A. Therefore, the first set is not closed.
- The second set is open, as it is $U \cap X$ with $U = (-\infty, -\frac{1}{2}) \cup (\frac{1}{2}, \infty)$ open in $\mathbf{R}$, but it is not closed, as its complement $[-\frac{1}{2}, \frac{1}{2}] \subset X$ cannot be written as $U \cap X$ for an open set $U \subset \mathbf{R}$ as such U would contain $\frac{1}{2}$ and therefore $\frac{1}{2} + \epsilon$ for some $\epsilon > 0$.
- The third set is not open as any open set of $\mathbf{R}$ containing $\frac{1}{2}$ also contains $\frac{1}{2} - \epsilon$ for some $0 < \epsilon < 1$. It is also not closed for the exact same reason that the first set is not closed.
- The fourth set is not open for the exact same reason that the third is not open, but it is closed as its complement $(-\frac{1}{2}, \frac{1}{2})$ in X is open as a subset of $\mathbf{R}$.

Note. With a bit more theory we could have avoided checking for closedness for the first two subsets: you will see that X is a ***connected*** *topological space, which implies that a nonempty proper subset cannot be both open and closed.*