TOPOLOGY SPRING 2024
SOLUTIONS SERIE 10

(1) (a) as $\mathscr{T}^*$ is defined from a basis, it suffices to check that the latter is closed under finite intersections (as clearly $\mathbf{R} \in \mathscr{B}_1$ and $\emptyset$ is obtained by a empty union), i.e. that both $\mathscr{B}_1$ and $\mathscr{B}_2$ are. We know it for $\mathscr{B}_1$ as it is a basis for the euclidean topology; the same is true for $\mathscr{B}_2$ as the intersection of two of its elements is the intersection of two elements of $\mathscr{B}_1$ minus the set B, i.e. an element of $\mathscr{B}_1$ minus B, so an element of $\mathscr{B}_2$.
As the euclidean topology, generated by $\mathscr{B}_1$, is Hausdorff, and $\mathscr{T}^*$ is finer than that, it is also Hausdorff;

(b) A is closed in the euclidean topology and so it is in $\mathscr{T}^*$, while $\mathbf{R} \setminus B$ is $(-\infty, 0) \cup ((-1, 1) \setminus B) \cup (1, \infty)$, with the second set belonging to $\mathscr{B}_2$ and the first and third to the euclidean topology, and hence to $\mathscr{T}^*$; therefore, B is closed;

(c) as $U \in \mathscr{T}^*$, there are euclidean open intervals U_i, $i \in I$ in $\mathbf{R}$ such that $U = (\bigcup_I U_i) \cap T$ with either $T = \mathbf{R}$ or $T = \mathbf{R} \setminus B$. As $0 \in U$, there is $i \in I$ such that $0 \in U_i$, which implies the claim;

(d) it suffices to take $n > b^{-1}$;

(e) $1/n \in V$ as $B \subset V$; hence, the same argument as for a, b gives $c' < 1/n < d'$ so that $(c', d') \setminus B \subset V$. Therefore, setting $c = \max(c', \frac{1}{n+1})$, $d = \min(d', \frac{1}{n-1})$ gives $(c, d) \subset V$;

(f) as $c < 1/n$ and $c \geq \frac{1}{n+1}$, any $x \in (c, 1/n)$ works;

(g) we have $x \in V$ by e) and f). As also $x \notin B$ and $0 < x < 1/n$ by f), we get $x \in U$ by c) as $1/n < b$ by d);

(h) B is a closed set not containing 0, but we just proved that any two opens U, V containing 0 and B respectively must intersect, so $\mathbf{R}$ with the topology T^* is not regular, and hence not normal.

(2) (a) If $\phi(x) = \phi(y)$ then for any continuous $f : X \longrightarrow [0, 1]$ we have $f(x) = f(y)$, which implies $x = y$ as the normality of X guarantees that we can otherwise find such f with $f(x) = 0, f(y) = 1$;

(b) we verify this on the cofinite basis as usual: the preimage of some of open in it is $\{x \in X : f_1(x) \in U_1, ..., f_n(x) \in U_n\}$ for some f_i's in $\mathscr{F}$ and $U_i \subset [0, 1]$. But this is just the intersection of the $f_i^{-1}(U_i)$, which is open;

(c) such a FSN is given by the intersection of a FSN for y in the product space with Y. The former is given, in virtue of the definition of product topology, (for example) by the usual cofinite neighborhoods $\{w \in \prod_{\mathscr{F}} X_f : |w_{f_j} - y_{f_j}| < \epsilon_j\}$ for all $j \in J$, as J and the ϵ_j's vary as in the text of the exercise. As we intersect with $\phi(X)$ we have $w = \phi(z)$ for some $z \in X$, so that $w_{f_j} = f_j(z)$ by definition, from which the claim follows;

(d) the existence of such V is granted by the "3bis" of the equivalent definitions of normality we saw, applied to $\{x_0\}$ and U^c. The existence of the map is implied by the first of the equivalent definitions, that gives us a map such that $g(U^c) = 0$ (and $g(x_0) = 1$);

(e) we just need to show that $\phi : X \longrightarrow Y$ is open: let $y = \phi(x_0) \in \phi(U)$ for some $U \subset X$ open, we need to show that $\phi(U)$ contains some set as those defined in point c). Then taking g as in the previous point we have $y_g = g(x_0) = 1$ and $\{|g(x) - g(x_0)| < 1/2\} \subset U$, so we are done;

(f) as $\prod_{\mathscr{F}} X_f$ is compact by Tychonoff's Theorem, we are done by the previous point.

(3) (a) Let us prove the base step $j = 1$: we have that the complement C_1 of $U_2 \cup ... \cup U_k$ is a closed set contained in U_1 (as the U_i's form a covering). Therefore, $C_1 \cap U_1^c = \emptyset$, and, as X is normal, there are disjoint open sets $V_1 \supset C_1$, $Z_1 \supset U_1^c$, which means precisely that $\overline{V_1} \subset U_1$. As $C_1 \subset V_1$, the latter satisfies the required condition. The inductive step is the same of the base step: we start from on open covering and we replace one of the sets, which the above argument shows that *we can choose arbitrarily*, with a smaller open whose closure is contained in the previous one. Doing it k times on the U_i's, we are done;

(b) first, we can extract W_i from V_i as we did V_i from U_i preserving the covering condition. As X is normal and $\overline{W_i} \subset V_i$, we know that there are functions $g_i : X \longrightarrow [0,1]$ such that $g_i(\overline{W_i}) = 1$ and $g_i(X \setminus V_i) = 0$, so we are done;

(c) as $g_i(X \setminus V_i) = 0$ and $\overline{V_i} \subset U_i$ we have $\mathrm{Supp}(g_i) \subset \overline{V_i} \subset U_i$. As for any $x \in X$ there is $1 \leq i \leq k$ such that $x \in W_i$, by the first property of the previous point we get $\sum_{1 \leq i \leq k} g_i(x) \geq 1 \ \forall x \in X$;

(d) let $\Sigma = g_1 + ... + g_k$ and $f_i = \frac{g_i}{\Sigma}$. By the previous point we have $\Sigma(x) \neq 0 \ \forall x \in X$, so the f_i's are well defined and satisfy the support condition. By definition $\sum_{1 \leq i \leq k} f_i(x) = 1 \ \forall x \in X$, so $(f_i)_{1 \leq i \leq k}$ is our partition of unity subordinate the the covering $(U_i)_{1 \leq i \leq k}$.

(4) (a) Such a covering without the finiteness condition exists by definition of manifold. As X is compact we can extract a finite subcovering, so we are done;

(b) it exists because X satisfies the hypotheses of the previous exercise, as you saw in class that any compact Hausdorff space is normal.
$f_i\phi_i$ is continuous on U_i as the product of two continuous functions. As $\mathrm{Supp}(f_i) \subset U_i$ we have that for any open set $U \subset X$, $U = (U \cap U_i) \cup (U \cap (\mathrm{Supp}(f_i))^c)$, so we just need to prove that $g_i|_{\mathrm{Supp}(f_i))^c}$ is continuous, but by definition g_i is 0 on this open, so we are done;

(c) since for any $x \in X$ there is $1 \leq i \leq k$ such that $f_i(x) \neq 0$ by definition of partition of unity, if $\phi(x) = \phi(y)$ then there is i such that $f_i(x) = f_i(y) \neq 0$, so $x, y \in \mathrm{Supp}(f_i) \subset U_i$. But then having $g_i(x) = g_i(y)$ and $f_i(x), f_i(y) \neq 0$ implies $\phi_i(x) = \phi_i(y)$, which, as ϕ_i is in particular bijective, implies $x = y$;

(d) ϕ is continuous as the preimage of an open set is the intersection of the finitely-many preimages of each component, which are open sets as each component is continuous. Moreover, this implies that ϕ maps compact sets into compact sets; as a closed subset of X is compact, and a compact subset of $\mathbf{R}^N$ is closed, we get that ϕ is closed, and hence open on its image. Being injective, this implies it is a homeomorphism on the image.