TOPOLOGY SPRING 2024
SOLUTIONS SERIE 11

We denote by I the interval $[0,1]$

(1) (a) If $X = A \cup B$ with A, B nonempty disjoint open sets, then taking $x \in A$, $y \in B$ and a path γ from x to y we'd have that $I = \gamma^{-1}(A) \sqcup \gamma^{-1}(B)$, a disjoint union of nonempty (as $0 \in \gamma^{-1}(A),\ 1 \in \gamma^{-1}(B)$) open sets, which is absurd as I is connected;

(b) symmetry is given by the constant path (which is always continuous), and if γ is a path from x to y then $\gamma \circ h$ is a path from y to x where $h : I \longrightarrow I$ is given by $t \mapsto 1 - t$. Finally, if γ is a path from x to y and γ' is a path from y to z then the path $\eta = \begin{cases} \gamma \circ d & \text{if } t \leq 1/2; \\ \gamma' \circ (d-1) & \text{if } t \geq 1/2; \end{cases}$
with $d : I \longrightarrow I,\ t \mapsto 2t$ is a well defined - as both branches are continuous and give y for $t = 1/2$ - path from x to z (formulae aside, we simply "glued" the two paths);

(c) the equivalence class of x is path-connected by definition, so it is connected by a). Again by (well-)definition of connected component, it thus must be contained in the connected component of x;

(d) let C be the path-connected component of x and let $y \in C$. We know that there is a homeomorphism $\phi : U \longrightarrow V$ from some open neighborhood of y to some open $V \subset \mathbf{R}^n$. Let $w = \phi(y)$; then V contains an open n-disk D centered at w (as those form a FSN), so ϕ restricts to a homeomorphism from $U_1 = \phi^{-1}(D)$ to D. As D is path-connected ($\mathbf{R}^n$ is locally path-connected), U_1 is. Therefore, as $y \in U_1$ by construction, $U_1 \subset C$ by b), hence we have proven that given $y \in C$, C contains an open neighborhood of y, and hence is open;

(e) X being contractible means precisely that Id_X is homotopic to a constant map x_0 on X. But then there is $F : X \times I \longrightarrow X$ continuous so that $F(x,0) = x,\ F(x,1) = x_0\ \forall x \in X$; therefore, $\gamma_x(t) = F(x,t)$ is, for any $x \in X$, a path from x to x_0. Again by b), this means that x is path connected, as to get a path from x to y we can join one from x to x_0 with one from x_0 to y.

(2) Let $P = (0,1)$.

(a) Observe that $C - \{P\}$ is connected (it is a comb with teeth at abscissae $1/n$) and that any open disk centered at P intersects $C - \{P\}$: if it has radius ϵ, it contains $(1/N, 1)$ for $N > \epsilon^{-1}$. Therefore, if C was the union of disjoint nonempty open sets, both would intersect $C - \{P\}$, making it also disconnected, which is absurd;

(b) $\{P\}$ is closed in C as it is closed in the euclidean topology, so its preimage is closed, and nonempty as it contains 0;

(c) V is open in C as its intersection with an open of $\mathbf{R}^2$ given by the same conditions. By the definition of continuity, as $\gamma(t_0) = P \in V$, there is an open $U \ni t_0$ such that $\gamma(U) \subset V$. As open intervals are a basis for the 1D euclidean topology, we can find an interval neighborhood of t_0 which is contained in U;

(d) it is as the continuous image of a connected set;

(e) let $\gamma((a,b)) = G$. As $|y-1| < \epsilon < 1/2$, G does not intersect the real axis. So if we had $(1/n, u) \in G$ for some $n \geq 1,\ u \leq 1$ (i.e. $G \neq \{P\}$), we could write $G = (G \cap D) \sqcup (G \cap \overline{D}^c)$ as the disjoint union of two nonempty open subsets - contradicting the previous question - where D is the open disk centered at $(1/n, u)$ of radius $\frac{1}{2n}$. Indeed D which does not contain P but $\overline{D}^c$ does because of the radius length;

(f) we just proved that if $t_0 \in Y$ then Y contains an open neighborhood of t_0, so it is open;

(g) being also nonempty and closed, Y must then be the whole space I as the latter is connected.

(3) (a) Symmetry follows from the symmetry in X, Y of the definition, and reflexivity is given by the identity map. Moreover, if X and Y have the same homotopy type via f, g as in the text and Y and Z too via h, l, we get that $f_1 = h \circ f : X \longrightarrow Z$ and $g_1 = g \circ l : Z \longrightarrow X$ are maps such that $g_1 \circ f_1$ is homotopic to Id_X and $f_1 \circ g_1$ is homotopic to Id_Z, as we can perform just the homotopy that brings $l \circ h$ to Id_Y for $t \in [0, 1/2]$ by doubling the speed, and the one bringing $g \circ f$ to Id_X for $t \in [1/2, 1]$, obtaining an homotopy from $g_1 \circ f_1$ to Id_X (they glue because at $t = 1/2$ we have reached an homotopy equivalence between $l \circ h$ and Id_Y, so that the map $Y \longrightarrow Y$ in the triangular diagram with f and g is precisely the identity, allowing us to work with $g \circ f$). The other direction is analogous;

(b) suppose we have two maps g_1, g_2 with that property. As we saw that we can compose homotopies, we have:

$$g_1 \sim g_1 \circ Id_Y \sim g_1 \circ (f \circ g_2) \sim (g_1 \circ f) \circ g_2 \sim Id_X \circ g_2 \sim g_2;$$

(c) we proved this in a);

(d) X has the homotopy type of $\{x_0\}$ iff there exists a map $g : \{x_0\} \longrightarrow X$ such that $g \circ f \sim Id_X$ where f is the only map from X to $\{x_0\}$. But then any such $g \circ f$ is exactly a constant map $X \longrightarrow X$, and we get precisely the definition of contractible space;

(e) in the first case, given a map $f : Y \longrightarrow X$ we can write $f \sim f \circ Id_X \sim f \circ f_0$ where $f_0 : X \longrightarrow X$ is a constant map, as we proved that X is contractible, giving the required homotopy from f to a constant map $Y \longrightarrow X$. In the other case we do the same thing but by composing the identity - and deforming it to a constant map - to the left of a map $g : X \longrightarrow Y$.

(4) (a) Let i, ϕ respectively be the inclusion and the group morphism of the text. For any $y_0 \in Y$, $r : X \longrightarrow Y$ induces another group morphism $\psi : \pi_1(X, y_0) \longrightarrow \pi_1(Y, y_0)$, and since $r \circ i = Id_Y$ we have that $\psi \circ \phi$, which is the morphism it induces on $\pi_1(X, y_0)$ by functoriality, is also the identity, so ϕ must be injective;

(b) we saw this in other terms on a previous sheet too: the projection of $x \in \mathbf{R}^n - \{0\}$ to S_{n-1} is continuous (the preimage of a small open "circle" $\{d(z_0, z) < \epsilon\}$ in S_{n-1} is the open cone with tip in 0 it defines) and fixes the sphere, so it's a retraction;

(c) a) and b) give an injective homomorphism $\mathbf{Z} \simeq \pi_1(S_1, y_0) \hookrightarrow \pi(\mathbf{R}^2 - \{0\}, y_0)$, so the latter cannot be trivial;

(d) if it was, we'd get an injective homomorphism $\mathbf{Z} \longrightarrow \{1\}$, which is absurd.

(5) (a) g is the composition $(T \circ (f, Id))(z) : D \longrightarrow S_1$ where $T : D^2 \setminus \Delta \longrightarrow S_1$ is the map sending two distinct points in D to the intersection of the line between them with S_1 (and Δ is the diagonal). T is open, as given (x_0, y_0) with $T(x_0, y_0) = P$ and an open neighborhood $U \ni P$ in S_1, there is $\epsilon > 0$ such that the circular sector $(P - \epsilon, P + \epsilon)$ (in radians, say) lies in U, and we can definitely find $\delta > 0$ such that for x and y in a open balls of radii δ around x_0 and y_0 respectively, the line between x, y intersects the above circular sector (as we can bound the distance of its intersection with S_1 from that of the line between x_0 and y_0 linearly in terms of δ and the diameter of the circle - simply by definition of what a "line" is). Obviously $(f, Id) : D \longrightarrow D^2$ is continuous as both component are, and its image is indeed contained in $D^2 \setminus \Delta$ by the no-fixed-point hypothesis, so g is well defined and continuous as the composition of continuous maps;

(b) above;

(c) having a continuous map $D \longrightarrow D$ which is the identity on S_1, we get that the latter is a retract of the former, so by Exercise 4 we get an injective morphism $\mathbf{Z} \simeq \pi_1(S_1) \hookrightarrow \pi_1(D) \simeq \{1\}$ (the last isomorphism as D is contractible), which is absurd, so f must have a fixed point.