TOPOLOGY SPRING 2024
SOLUTIONS SERIE 12

(1) (a) It suffices to notice that all points have a path to x_0 as it lies in all circumferences and any point belongs to one of the circumferences;

(b) such a neighborhood contains the intersection of X with an open ball $B(0,\epsilon)$ of $\mathbf{R}^2$ centred at the origin, so it contains C_n for $n > \frac{2}{\epsilon}$;

(c) let $U \subset C_n$ be open. If $x_0 \notin U$ then $r_n^{-1}(U) \simeq U$ via the identity; otherwise, $r_n^{-1}(U) = U \cup (X \setminus C_n) = V$, so we need to prove that this union is open in X. Given any $P \in C_n^c \cap V$, V contains any of neighborhood of P contained in its circle, whereas for $P \in C_n \cap V$ we can take U;

(d) as r_n is a retraction, the map $r_n \circ i : C_n \hookrightarrow X \longrightarrow C_n$ is the identity, so the induced map on fundamental groups must be too, and in particular the last arrow is surjective;

(e) it is well defined as for any t there is exactly one such n and $\gamma(0) = \gamma(1) = x_0$. γ is continuous on $[0,1)$ as each "segment" is (they are linear scalings of continuous loops) and substituting the extremes of the interval gives $\gamma_n(0) = \gamma_{n-1}(1) = x_0$. Finally, it is also continuous at $t = 1$ as any neighborhood of x_0 contains a C_n by b), and hence its preimage contains $(t, 1]$ for $t > 1 - 1/n$;

(f) r_{n*} sends the class of γ to the class of the loop that is constant for $t < 1-1/n$, then γ_n at $n(n+1)$-times the speed, and then constant again. Linearly changing, from 0 to $1 - 1/n$ the point (and consequently the speed) of when the loop starts following γ_n clearly describes an homotopy of it with γ_n;

(g) we define the morphism as r_{n*} component wise, and the previous point precisely tells us that given any element in the image, that is a collection $\{[\gamma_n], n \geq 1\}$ with γ_n a loop at x_0 in C_n, there is a class in the domain mapping precisely to that, so the morphism is surjective. As a countably infinite product of copies of $\mathbf{Z}$, $\pi_1(X, x_0)$ has therefore the cardinality of the continuum.

(2) (a) Let

$$D = \{\delta \in (0,1] : \exists m \geq 1 \text{ and } 0 = t_0 < ... < t_m = \delta \text{ with the property of the text}\}$$

and let us show that D is clopen, which implies the statement. Surely D is nonempty and open, as if $\gamma(t) \in U_i$ then there is $\epsilon > 0$ such that $\gamma([t, t+2\epsilon)) \subset U_i$, and hence $\gamma([t, t+\epsilon]) \subset U_i$, by continuity of γ. Also D is an interval by construction, so if it is not closed, there is $a \leq 1$ such that $D = (0, a)$. This means that for any $\epsilon > 0$ and for all $a - \epsilon < b < a$ there are $m \geq 1$ and time intervals $0 = t_0 < ... < t_m = b$ with corresponding $\{U_{i(k)}, 0 \leq k < m\}$ satisfying the hypotheses with an index $i = i(m-1) \in I$ such that $\gamma([a - \epsilon, b]) \subset U_i$, but $\gamma(a) \notin U_i$. Then, since the U_i's cover X, there is U_j such that $\gamma(a) \in U_j$; but U_j is also open, so there is $\mu > 0$ such that

$\gamma([a-\mu, a+\mu]) \subset U_j$ by continuity as before. Therefore $a \in D$, since we can take $b \in (a-\mu, a)$ with the respective sequence of $t_k, 0 \leq k \leq m$ (U_j did not depend on b), and define $t_{m+1} = a$ and $U_{i(m)} = U_j$;

(b) we know that $x_0 \in U_i \; \forall i \in I$ and that for $0 \leq k < m$, $\gamma(t_{k+1}) \in U_{i(k)} \cap U_{i(k+1)}$ with these intersections being path-connected, so for such k there is a path β_k from x_0 to $\gamma(t_k)$ contained in the intersection. It is then natural to define γ_k as follows:

$$\gamma_k(t) = \begin{cases} \beta_k(3t) & \text{if } 0 \leq t \leq 1/3 \\ \gamma(t_k + 3(t-1/3)t_{k+1}) & \text{if } 1/3 \leq t \leq 2/3 \\ \beta_{k+1}^{-1}(3t-2) & \text{otherwise;} \end{cases}$$

then these are well-defined loops at x_0 contained in $U_{i(k)}$, and they link to a path homotopic to γ, as the "links" $\beta_k^{-1}(3t-2) \cdot \beta_k(3t)$ are homotopic to the constant loop by definition of inverse, and the rest of the loop is already constructed as piece-wise homotopic to m paths that link to γ;

(c) given a loop γ at x_0, the previous point assures us that we can write γ as the composition of m loops γ_k at x_0 each contained in one of the open sets U_i, which have trivial fundamental group. Therefore each of the γ_k is homotopic to the constant loop in $U_{i(k)}$, and hence in X, and so then must be γ.

(3) Take two distinct points $x, y \in \bigcup_{i \in I} A_i$, so let us say $x \in A_j, y \in A_k$; moreover, there exists $z \in \bigcap_I A_i$, so $z \in A_j \cap A_k$. As each A_i is path-connected, there exist paths from x to z and from z to y, and hence a path from x to y.

(4) (a) $S_2 \setminus \{p\}$ and $S_2 \setminus \{q\}$ are path-connected as they are homeomorphic to the plane via stereographical projection from the missing "pole". As their intersection $S_2 \setminus \{p, q\}$ is nonempty (it contains $(0, 1, 0)$), the previous Exercise immediately gives that S_2 is path-connected. Then $S_2 \setminus \{p, q\}$ must be too, as given two points x,y in it there is a path γ in S_2 joining them, which we can take so that it does not contain any of the poles: for example, let U be a circular open neighborhood of p not containing x and y (S_2 is Hausdorff), then if $Im(\gamma) \cap U \neq \emptyset$ and $0 \leq u < v \leq 1$ are such that $\gamma(u), \gamma(v)$ are respectively the first and last intersection of γ with ∂U (which exist as γ is continuous and $x, y \notin U$), then we can modify γ into a loop from x to y disjoint from U as follows: for $0 \leq t \leq u$ follow γ, then for $u \leq t \leq v$ follow (say, clockwise) ∂U in such a way that you are at $\gamma(v)$ at time v, and then follow γ again. The same process can be then done if necessary for a neighborhood V of q which can be taken as to not intersect ∂U so that the changes don't interfere, and we are done;

(b) both spaces are homeomorphic to the plane, which has trivial fundamental group;

(c) this follows directly from Exercise 2c), as $S_2 = (S_2 \setminus \{p\}) \cup (S_2 \setminus \{q\})$ and we proved that $S_2 \setminus \{p\}$ and $\mathrm{S}_2 \setminus \{q\}$ have trivial fundamental group and that their intersection(s) are path connected.