TOPOLOGY SPRING 2024
SOLUTIONS SERIE 13

(1) (a) By definition, there exists a neighborhood U of $p(y)$ such that $p^{-1}(U)$ is isomorphic to $U \times D$ for D a discrete set, via the diagram seen in class. So y is contained in an open V isomorphic to $\{d\} \times U$ for some $d \in D$, and the restriction of f to V gives the required isomorphism;

(b) as X is connected and fibers are finite, it suffices to show that for any integer $k \geq 1$, the set $F_k = \{x \in X : |p^{-1}(x)| = k\}$ is open: indeed, if that is the case, all these sets will also be closed, as $F_{k_0} = X \setminus (\bigcup_{k \neq k_0} F_k)$, and hence either empty or equal to X. As they are disjoint and cover X, we will have $X = F_k$ for some k, which is the thesis.
Openness follows easily from the definition: if $x \in F_k$, as we know that there is a neighborhood U of x and a discrete set D such that $f^{-1}(U) \simeq U \times D$ in a commutative diagram where $U \times D$ maps to U via the canonical projection, every point in U has fiber in bijection with D, and therefore of the same cardinality of that of x.

(2) (a) All axioms defining a group action are symbolically satisfied as the identity acts trivially and the action of a composition is given by composing the permutations at the index;

(b) for $n = 1$ the action is trivial and hence gives the identity covering. For $n \geq 2$, the fiber above each point has finitely many elements: any equivalence class is a S_n-orbit and hence has at most $n!$ elements. Moreover $\mathbf{C}^n$ is connected, and we have fibers of different cardinality: above the image of $(x_1, x_2, ..., x_n)$ with $x_1 \neq x_2$ we have both $(x_1, x_2, ..., x_n)$ and $(x_2, x_1, ..., x_n)$, whereas above the diagonal $(x, x, ..., x)$ we have fibers with one element. These things together yield that p cannot be a covering by Exercise 1;

(c) permuting n distinct numbers gives n distinct numbers, so $\sigma U_n = U_n$. Let p be the projection, $X = U_n/S_n$, $x \in X$ and $y_1, ..., y_{n!}$ its (distinct!) preimages. Then, as $\mathbf{C}^n$ is Hausdorff, we can chose open neighborhoods V_i of y_i that are disjoint for $i \neq j$. So $U = \bigcap_{i=1}^{n!} p(V_i)$ is an open neighborhood of x, whose preimage is isomorphic to $U \times \{1, ..., n!\}$ precisely via the action of S_n, making the relative diagram as seen in class commute.

(3) (a) The fibers of f are finite, as over a field the equation $f(x) = a$ can have at most $d = \deg f$ distinct solutions. So, by Exercise 1, if f was a covering, all points would have fibers with the same cardinality: let us prove that for $d \geq 2$ this is never the case. As a polynomial has distinct roots over $\mathbf{C}$ precisely when it has no common roots with its derivative (this is a direct consequence of the formula for the derivative of the product), we just need to show that there are $a, b \in \mathbf{C}$ such that $f - a$ has a common root with f' and $f - b$

doesn't (the derivatives of polynomials of the form $f -$ constant are all f'). Let $\alpha_1, ..., \alpha_{d-1}$ be the (possibly not distinct) roots of f'. Then saying that $f - a$ has α_i as a root is the same as saying $f(\alpha_i) = a$, i.e. $a \in C_f$. As this set has between 1 and $n-1$ elements, we are done;

(b) it is an elementary fact of complex analysis that a function f holomorphic on some open set U and whose derivative at $z \in U$ is nonzero, is locally injective around z. Since polynomials are holomorphic this applies, so by restricting to some closed ball in the neighborhood around z where f is injective we get a neighborhood K_z as in the hint: we are left to prove that f is a closed map on K_z (as we can then take as V_z its inner part). Since f sends compact sets to compact sets by continuity, our K_z is compact, and a closed subset of a compact set is compact, we get that f restricted to K_z sends closed subsets to compact subsets, which are closed as $\mathbf{C}$ is Hausdorff;

(c) we showed the part of the hint about the size of fibers in the first point. By the second one, we can find neighborhoods V_{z_0} of the elements z_0 of the fiber over w_0 over which f is a homeomorphism. Taking as usual the image of their intersection, we get the desired trivialization.

(4) (a) we know that the projections from the cartesian product are continuous. As the fibre product has the subspace topology, p_1 and p_2 are simply their restrictions, and hence also continuous. Given $(x, y) \in Y'$, the identity $g \circ p_1 = p \circ p_2$ is equivalent to $g(x) = p(y)$, which is the equation describing Y';

(b) the map is continuous as a map to a product space whose coordinate-maps are continuous (one is the identity and the other one is the projection $X \times D \longrightarrow D$), bijective because for any $x \in X'$ there is $y \in Y$ such that $g(x) = p(y)$ as p is surjective, and since $y = (v, d_0)$ for some $v \in X, d_0 \in D$, we have $p((v, d)) = v = g(x)\ \forall d \in D \implies (x, (v, d)) \in Y'\ \forall d \in D$ by definition of Y'. The inverse map $\Psi^{-1} : (x, d) \mapsto (x, (v_d, d))$ is well-defined as given $x \in X', d \in D$, if there were $v_1 \neq v_2$ satisfying $g(x) = p((v, d))$ we'd have that the projection restriction $X \times D \supset X \times \{d\} \longrightarrow X$ wouldn't be a injective (and in particular a homeoomorphsim). It is continuous component-wise by the same argument for Ψ (but using the projection $X \times D \longrightarrow X$), and so continuous, hence Ψ is a homeomorphism;

(c) for a general covering p, we have an open cover $(U_i)_{i \in I}$ of X such that $p_i := p|_{p^{-1}(U_i)}$ is a trivial covering for all $i \in I$, giving a corresponding collection of homeomorphisms $\Psi_i : (p^{-1}(U_i))' \longrightarrow g^{-1}(U_i) \times D_i$ thanks to b). The $g^{-1}(U_i)$'s then form a cover of X' that gives the required local trivialization;

(d) by definition of the fibre product we have $p_1^{-1}(\{x\}) \longleftrightarrow \{y \in Y : p(y) = g(x)\} = p^{-1}(g(x))$.

(5) (a) The fiber of the pullback above $x \in S_1 = X'$ is $\{(x, y) : y \in S_1 = Y : y^n = x^n\} = \{\zeta_n^k x, k = 0, ..., n-1\}$ by identifying the two copies of S_1, where ζ_n is any primitive n-th root of unity. Therefore, $p_1 : Y' \longrightarrow X'$ is isomorphic to the projection $X' \times D \longrightarrow X'$ for $D = \{1, \zeta, ..., \zeta^{n-1}\}$ a discrete set of n elements;

(b) by definition of p_1 we have $p_1 \circ q(z) = z^2 = f_2(z)$.