TOPOLOGY SPRING 2024
SOLUTIONS SERIE 2

Given a metric space (M, d) we use the notation $B(x, r)$ for the subset $\{y \in M : d(x, y) < r\}$, where both M and d are going to be left implicit if there is no ambiguity.

(1) (a) Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be the topologies defined by d_1 and d_2. It clearly suffices to show that the first inequality implies that $\mathscr{T}_1 \subset \mathscr{T}_2$, i.e. that an open set for d_1 is open also for d_2. Let $\emptyset \neq U \in \mathscr{T}_1$ and $x \in U$: we want to prove that there exists $\epsilon > 0$ such that $B_{d_2}(x, \epsilon) \subset U$; by hypothesis, there exists $\delta > 0$ such that $B_{d_1}(x, \delta) \subset U$, and by the first inequality we have $B_{d_2}(x, r) \subset B_{d_1}(x, r^a)$ for any $r \geq 0$, so we get the desired claim for $\epsilon = \delta^{a^{-1}}$;

(b) $\delta(x, y)$ is clearly non negative and is 0 exactly when the numerator $d(x, y)$ is 0, so when $x = y$. Moreover, it is symmetric as it depends only on d, which is symmetric. Finally, we have:

$$\begin{aligned}\delta(x, y) + \delta(y, z) &= \frac{d(x, y) + d(y, z) + 2d(x, y)d(y, z)}{1 + d(x, y) + d(y, z) + d(x, y)d(y, z)} \geq \\ &\geq \frac{d(x, y) + d(y, z) + 2d(x, y)d(y, z)}{1 + d(x, y) + d(y, z) + 2d(x, y)d(y, z)} = \\ &= \left(1 + \frac{1}{d(x, y) + d(x, z) + 2d(x, y)d(y, z)}\right)^{-1} \geq \\ &\geq \left(1 + \frac{1}{d(x, y) + d(y, z)}\right)^{-1} \geq \\ &\geq \left(1 + \frac{1}{d(x, z)}\right)^{-1} = \delta(x, z)\end{aligned}$$

Clearly $\delta(x, y) \leq d(x, y)$ for all x, y so $\mathscr{T}_\delta \subset \mathscr{T}_d$ by the proof of point a) with $a = 1$. So let us prove the other inclusion: let $U \in \mathscr{T}_d$ and let $x \in U$, for which we know there exists $\epsilon > 0$ such that $B_d(x, \epsilon) \subset U$. Observe that the function $t \mapsto \frac{t}{1+t}$ is continuous and monotonically increasing in a right neighborhood of 0, so for small enough ϵ there exists $\gamma = \gamma(\epsilon)$ such that $f(t) < \gamma \iff t < \epsilon$, and in particular $\gamma = \frac{\epsilon}{1+\epsilon}$. So we obtain that $B_d(x, \epsilon) = B_\delta(x, \gamma)$, which gives the desired inclusion. Clearly $\delta(x, y) < 1$ as $d(x, y) < 1 + d(x, y)$.

(2) (a) δ is clearly nonnegative, and for it to be 0 we must be in the first case (as otherwise one of x and y is nonzero and so $\delta(x, y)$ is positive), so it is zero precisely when d is. Symmetry also immediately follows from that of d. Let us verify the triangle inequality: if x, y, z are all proportional, we are in the first case and the result is simply the triangle inequality for d; otherwise either x and y or y and z are not proportional, say WLOG $x \notin \mathbf{R}y$, so we have $\delta(x, z) \leq d(x, 0) + d(0, z) \leq d(x, 0) + d(0, y) + d(y, z) = \delta(x, y) + \delta(y, z)$, where the first inequality follows from the definition and the triangle inequality for

d applied to the triple $(x, 0, y)$, and the second from the triangle inequality for d applied to $(0, y, z)$;

(b) Let $P = (x_0, y_0)$. If $P = 0$ is the origin then $B_\delta(0, r)$ is the open disk of radius r and centre 0, as any point is proportional with 0; otherwise, $B_\delta(P, r)$ consists of the open segment (i.e. with the extrema removed) of centre P and length $2r$ lying on the line between P and the origin, along with the open disk of radius $\max(0, r - \sqrt{x_0^2 + y_0^2})$ centred at the origin;

(c) This again follows by the fact that $d_{\text{eucl}} \leq \delta$ and exercise 1 a);

(d) Thanks to point b) we know that for example $B((1, 1), 1)$, which is open for δ, is an open segment in $\mathbf{R}^2$, which is not open in the euclidean topology as balls are 2-dimensional.

(3) (a) As in the discrete topology every subset is open, the openness of singletons follows. In the other direction, let $U \subset X$ be any set, then $U = \cup_{x \in U}\{x\}$ is open as the union of open sets;

(b) Define $d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{else.} \end{cases}$

This is a distance: the only nontrivial property is the triangle inequality, which reduces to $0 \leq 0$ if $x = y = z$, to $1 \leq 1$ if $x = y \neq z$ or $x \neq y = z$, to $0 \leq 2$ if $x = z \neq y$ and to $1 \leq 2$ in the case in which they are three distinct points. With its induced topology the singletons are open as $x = B_d(x, r)$ for any $0 < r < 1$, and so $\mathscr{T}_d$ is the discrete topology by the previous point;

(4) (a) to verify that t is well defined we just need to check that $t((x_i)) \in [0, 1]\ \forall (x_i) \in C$ (the power series is clearly convergent as the coefficients are in $\{0, 1\}$); but from the geometric series formula (and the nonnegativity of the x_i) we have $0 \leq t((x_i)) \leq 2\sum_{n \geq 1} 3^{-n} = 2((1 - \frac{1}{3})^{-1} - 1) = 1$.

To show injectivity we can drop the factor of 2 and show that $t/2$ is injective. If we take $(x_i), (y_i)$ with $(x_i) \neq (y_i)$, there is the **smallest** index n where they differ, so say WLOG $x_n = 0, y_n = 1$. But then we will have, again from the geometric series formula,

$$t((y_i)) - t((x_i)) \geq (\frac{y_n}{3^n} - \frac{x_n}{3^n}) - \sum_{i \geq n+1} 3^{-i} = 3^{-n} - \frac{3^{-(n+1)}}{1 - 3^{-1}} = \frac{3^{-n}}{2} > 0.$$

Finally, let $V \subset [0, 1]$ be open: we need to show that for any sequence (x_i) in $U = t^{-1}(V)$ there is a finite set of indeces I such that U contains the sequences that match (x_i) for the indeces in I. Let $t((x_i)) = \alpha \in V$; then there is $\epsilon > 0$ such that $B(\alpha, \epsilon) \cap [0, 1] \subset V$ with the ball taken as a subset of $\mathbf{R}$ with the euclidean metric; the need for intersecting with $[0, 1]$ only comes from the case where $\alpha \in \{0, 1\}$, otherwise the ball would entirely be contained in $[0, 1]$. So we can suppose WLOG that $\alpha + [0, \epsilon) \subset U$ and $\alpha \neq 1 \iff x \neq (1, 1, 1, ...)$ (as this latter case has $\alpha - [0, \epsilon) \subset V$ and is equivalent to ours with $\alpha = 0$). But then, taking $N = N(\epsilon)$ such that $2\sum_{n > N} 3^{-n} < \epsilon$, so any $N > \log_3(\epsilon^{-1})$, implies that if $y_i = x_i$ for $i \in \{1, ..., N\}$ then $t((y_i)) \in V$;

(b) as the hint suggests, if $x \in [0,1] \setminus t(C)$ then we cannot write $x = \sum_{n\geq 1} \frac{b_n}{3^n}$ with $b_n = 2x_n \in \{0,2\}$ for any $(x_i) \in C$, but x has a base-$\frac{1}{3}$ expansion, which then has to be $x = \sum_{n\geq 1} \frac{a_n}{3^n}$ with at least one $a_n = 1$. Let us show that there is $\epsilon = \epsilon(x) > 0$ such that all $y \in B(x,\epsilon) \cap [0,1]$ share the same property: this will prove that the complement of $t(C)$ is open. As in the previous point, this boils down to the fact (which we proved above) that if n is the smallest index where two sequences in C differ, then the base-$\frac{1}{3}$ power series constructed from them differ of at least $3^{-n}/2$. Now, the sequence $(\frac{a_i}{2})$ is not in C precisely because $a_n = 1$, but the exact same argument proves that the power series constructed from it must differ from any with $x_i \in \{0,1\}\ \forall i$ of at least $3^{-n}/4$, and so $|x - t((x_i))| \geq \frac{3^{-n}}{2}\ \forall (x_i) \in C$ and hence $\epsilon = 3^{-n}/2$ works;

(c) to show that t^{-1} is continuous from the image to C we just need to prove that if $U \subset C$ is open then $t(U) \subset t(C)$ is open in the subspace topology. Let $x = t((x_i)), (x_i) \in U$. As U is open there is I finite such that the sequences in C that agree with (x_i) on I are in U. But by the argument of a) there is $\epsilon = \epsilon(I)$ such that all sequences (y_i) with $|t((y_i)) - x| < \epsilon$ must agree with (x_i) on I (just take $\epsilon = 3^{-\max(I)-1}$), so $B(x,\epsilon) \cap t(C) \subset t(U) \implies t(U)$ is open.