TOPOLOGY SPRING 2024
SOLUTIONS SERIE 3

(1) (a) $0 \in \bar{A} \iff \forall U$ open set containing 0, $U \cap A \neq \emptyset$. But such U necessarily contains the set $U_{(x_i),\epsilon} = \{g \in X : |g(x_i)| < \epsilon \ \forall i = 1, ..., N\}$ for some $(x_i)_{i=1}^N \in \mathbf{R}^N$ and $\epsilon > 0$ by definition, and all the $U_{(x_i),\epsilon}$ (as $(x_i), \epsilon$ vary) are clearly open sets containing 0, so $0 \in \bar{A} \iff \forall N, ((x_i), \epsilon) \in \mathbf{R}^N \times \mathbf{R}^+$ we have $A \cap U_{(x_i),\epsilon} \neq \emptyset$, which is precisely the claim;

(b) for A the set of indicator functions, clearly $0 \notin A$ as such a function vanishes only on a finite set by definition, but $0 \in \bar{A}$ as for any open neighborhood U of 0 there are N and $x_1, ..., x_N \in \mathbf{R}$ such that $\{g : \mathbf{R} \longrightarrow \mathbf{C} : g(x_i) = 0 \ \forall i = 1, , , ., N\} \subset U$, so $f_{\{x_i\}} \in U$ with the notation of the exercise test.
For A the monic polynomials, clearly $0 \neq A$, as it is not a monic polynomial; on the other hand, for any $N \geq 1$ and $(x_i) \in \mathbf{R}^N$ we can find a real monic polynomial p such that $p(x_i) = 0$ as the relative interpolation problem always has solution, so we are done by point a);

(c) Given a sequence $(f_n)_{n \geq 0}$, the set $Z = \{x \in \mathbf{R} : \exists n \geq 0 : f_n(x) = 0\} \subset \mathbf{R}$ is a countable union of finite sets, and therefore countable. But $\mathbf{R}$ is uncountable, so we can find $x \in \mathbf{R}$ where none of the f_n vanishes, which proves their sequence cannot converge to 0 in the pointwise convergence topology (as they equal 1 when nonzero).

(2) (a) Let $U \subset X$ be open and nonempty: we need to show that $A \cap U \neq \emptyset$. From the definition of the pointwise convergence topology we know that U contains some $U_{(x_i),\epsilon}(f) = \{g \in X : |g(x_i) - f(x_i)| < \epsilon \ \forall i = 1, ..., N\}$ for some $f \in X$ and $N \geq 1$, $(x_i) \in \mathbf{C}^N$, $\epsilon > 0$, so this is equivalent to showing that for any such f, N and $((x_i), \epsilon) \in \mathbf{C}^N \times \mathbf{R}^+$ there is a polynomial p such that $|p(x_i) - f(x_i)| < \epsilon \ \forall i = 1, ..., N$, but it is well known that the interpolation problem for N pairs in $\mathbf{C}^2$ has solution in polynomials of degree at most $N-1$, so we are done by applying it to the pairs $(x_i, f(x_i))$;

(b) if we had a polynomial p and a neighborhood $U \supset p$ (for $\mathscr{T}_p$) such that $U \subset A$ then we will have N points $(x_i) \in \mathbf{C}^N$ and an $\epsilon > 0$ such that all functions $f : \mathbf{C} \longrightarrow \mathbf{C}$ satisfying $|f(x_i) - p(x_i)| < \epsilon \ \forall i = 1, ..., N$ are polynomials. Clearly this is false as we can define f as matching p on the x_is and equal to 0 everywhere else except for $f(z) = 1$ at some other $z \neq x_i$, so f is discontinuous and hence not a polynomial;

(c) this follows easily from the definition: first of all, the collection $\{V_{f,n} : n \geq 1\}$ clearly is countable for f fixed. Second, each open set $U \supset f$ must contain $V_{f,n}$ for some n as by definition it contains $\{g \in X : |g(x) - f(x)| < \epsilon \ \forall x \in \mathbf{C}\}$ for some $\epsilon > 0$ and there is n such that $n^{-1} < \epsilon$. Finally, let us prove they are open: let $g \in V_{f,n}$. We know $\delta = \delta_g = \sup_{\mathbf{C}} |g(x) - f(x)| < n^{-1}$, so by the triangle inequality $V_{f,n}$ contains $\{h \in X : |g(x) - h(x)| < n^{-1} - \delta \ \forall x \in \mathbf{C}\}$ and therefore is open;

(d) we just need to prove that for any polynomial p with 0 constant term there is $n \geq 1$ such that $V_{p,n}$ does not contain other polynomials with 0 constant term. But any such n works as otherwise we would have two polynomials p, q whose difference is a bounded, and therefore constant, function by Liouville's Theorem (as polynomials are holomorphic), which means that $p = q$ as they agree for $z = 0$.

(3) (a) $\mathbf{C}^n$ is the zero-set of the polynomial 0, and the empty set is the zero-set of the polynomial 1. Given two algebraic sets with associated families of polynomials $(f_i)_I, (g_j)_J$, their union is the zero-set of the family $(f_i g_j)_{I \times J}$ (as $(f \cdot g)(z_i) = 0 \iff$ either $f(z_i) = 0$ or $g(z_i) = 0$; notice how this fails for arbitrary unions because polynomials have finite degree). Given algebraic sets $(A_s)_S$ with associated families of polynomials $F_s = (f_i^s)_{i \in I_s}$ (here s is an index, not an exponent), their intersection is the algebraic set corresponding to the family $\bigcup_{s \in S} F_s$, by definition. So we proved that the complements of the algebraic sets define a topology, the Zariski topology;

(b) we just need to show that the zero-sets of nonzero complex polynomials in one variable are all the finite subsets of $\mathbf{C}$. Surely they are finite as any nonzero polynomial over a field has finitely many roots in it (equal to the degree over an algebraic closure), and we can obtain any finite set $S = \{x_1, ..., x_n\}$ since the polynomial $\prod_{i=1}^n (x - x_i)$ vanishes precisely on S;

(c) let $A \subset \mathbf{C}^m$ be closed, and hence algebraic, the zero-set of (say) $(g_j)_J$. The preimage of A under a polynomial map f is $\{x \in \mathbf{C}^n : g_j(f_i(x)) = 0 \ \forall i = 1, ..., n, \ \forall j \in J\}$ and is therefore algebraic, relative to the family $(g_j \circ f_i)_{J \times \{1,...,n\}}$. So it is closed and hence f is continuous;

(d) it suffices (and is stronger) to show that the union of two algebraic sets C_1, C_2 different from $\mathbf{C}^n$ is never $\mathbf{C}^n$ (i.e. that any open set is dense). We show this in Exercise 4;

(e) say A is not dense, i.e. there exists a proper algebraic set containing A. Then there is a nonempty family $(f_i)_I$ of nonzero polynomials vanishing on A, and choosing any gives the desired claim. Notice how the two things are equivalent: if such f exists, A is contained in a proper algebraic set, and hence not dense;

(f) we argue by induction: for $n = 1$ the previous point tells us that if the claim is false then there exists a nonzero polynomial vanishing over infinitely many complex numbers, which is absurd. Assume now the thesis for $n - 1$: if the claim is false, we get a polynomial f with coefficients in $\mathbf{C}[x_1, ..., x_n] = \mathbf{C}[x_1, ..., x_{n-1}][x_n]$ vanishing on $\mathbf{Z}^n$, and hence, specializing $x_n = k$, polynomials $f_k \in \mathbf{C}[x_1, ..., x_{n-1}]$ vanishing on $\mathbf{Z}^{n-1}$, which by the previous point is absurd unless $f_k = 0$. But if $f_k = 0 \ \forall k \in \mathbf{Z}$ this means $f = 0$ (its coefficients as a polynomial in x_n have to be 0), which is absurd.

(4) (a) If U is not dense, by definition there is a Zariski nonempty open V such that $U \cap V = \emptyset$, so their complements A_1, A_2 are proper closed set whose union is $\mathbf{C}^n$;

(b) take $f \in I_1 \cap I_2$. Then f vanishes on A_1 and A_2 by definition, but we know $\mathbf{C}^n = A_1 \cup A_2$ and so f vanishes on all of $\mathbf{C}^n$, and therefore is 0 identically.

(c) working as the hint suggests, the product $f = f_1 f_2$ is in $I_1 \cap I_2$ but is nonzero as both the f_is are, which gives the desired contradiction.

(5) With the given identification, $GL_n(\mathbf{C})$ is the complement of the Zariski closed set $\det(x_1, ..., x_{n^2}) = 0$, where det is the degree-n^2 polynomial form that gives the determinant of a matrix when computed on its entries, hence it is open and therefore dense by the previous exercise (it contains the identity, so it is nonempty). So if any polynomial function of the entries vanishes on $GL_n(\mathbf{C})$ that means that $GL_n(\mathbf{C}) \subset C(f)$ with $C(f) = \{(x_1, ..., x_{n^2}) : f(x_i) = 0\}$ closed, but since $GL_n(\mathbf{C})$ is dense we have $C(f) = \mathbf{C}^{n^2}$ and so $f = 0$ vanishes on all matrices.