TOPOLOGY SPRING 2024
SOLUTIONS SERIE 4

(1) Let X be Hausdorff and $x \in X$. Moreover, let us refer to the intersection of the text as C_x. Surely $x \in C_x$; for any $y \neq x$ we have open neighborhoods U, V of x, y such that $U \cap V = \emptyset$, so $y \notin \overline{U} \Longrightarrow y \notin C_x \Longrightarrow C_x = \{x\}$.
Conversely, let $C_x = \{x\}\ \forall x$; this means that given $x \neq y \in X$ there is an open neighborhood U of x such that $y \notin \overline{U}$, and hence U and the complement V of $\overline{U}$ are open sets with $x \in U,\ y \in V$ and $U \cap V = \emptyset$, so X is Hausdorff by arbitrariness of x, y.

(2) (a) If $\mathscr{F}$ contained the neighborhood filters of $x, y,\ x \neq y$ then it would contain two disjoint open sets $U \ni x, V \ni y$, as we are assured of their existence by the Hausdorff property. But then it would contain their intersection $\emptyset$, which is absurd;

(b) if X is not Hausdorff there are $x \neq y \in X$ such that $V \cap W \neq \emptyset$ for all V, W neighborhoods of x, y respectively; then the set $\mathscr{F}$ in the hint is a filter as it does not contain the empty set by definition, is clearly closed by taking supersets as it is defined by a superset condition, and also closed by finite intersections as a finite intersection of neighborhoods is a neighborhood and $V \cap W$ is always nonempty; therefore, if it converges it has a unique limit. But $\mathscr{F}$ clearly converges to both x, y as we can choose $A = V \cap X = V$ and $A = X \cap W = W$.

(3) (a) Clearly $X \times Y$ and $\emptyset$ are open as we can see by taking $(V, W) = (X, Y)$ in one case and $(\emptyset, \emptyset)$ in the other (for any (x, y)). The union condition is clearly satisfied as if $U = \bigcup_I U_i$ and $(x, y) \in U$ then there is $i \in I$ such that $(x, y) \in U_i$ and we can take as (V, W) those that work for U_i. Finally, if $U = \bigcap_1^N U_i$ and $(x, y) \in U$, then $(x, y) \in U_i\ \forall i = 1, ..., N$ and there are (V_i, W_i) open in X, Y such that $(x, y) \in V_i \times W_i \subset U_i$, so setting $V = \bigcap_1^N V_i,\ W = \bigcap_1^N W_i$ then they are open in X, Y respectively and we have $(x, y) \in V \times W \subset U$;

(b) the diagonal being closed is equivalent to its complement being open, i.e. that for any $(x, y) \in X \times Y,\ x \neq y$, there are $V, W \subset Y$ open with $(x, y) \in V \times W$ and $V \times W \cap \Delta_X = \emptyset$ (it is clear that a basis for the product topology for $X \times Y$ in general is given by $\{V \times W,\ V \subset X,\ W \subset Y \text{ open sets}\}$). But this is precisely equivalent to V, W being disjoint, and so to having for any $x \neq y \in X$ disjoint open neighborhoods, which is precisely the Hausdorff condition;

(c) consider the map $\phi : X \times X \longrightarrow Y \times Y$ defined as $\phi(x, y) = (f(x), g(y))$. Then the first set is precisely $\phi^{-1}(\Delta_Y)$. But Δ_Y is closed by the previous point, and ϕ is continuous as the preimage of an open set $V \times W,\ V, W \subset Y$ of $Y \times Y$ is the open $f^{-1}(V) \times g^{-1}(W) \subset X \times X$ (we can clearly verify continuity on a base), so $\phi^{-1}(\Delta_Y)$ is closed. Similarly, the second set is once again just the preimage of the diagonal in $Y \times Y$ under the map $\psi : X \longrightarrow Y \times Y$, $\psi(x) = (f(x), g(x))$, which is continuous as the preimage of an open set $V \times W$, $V, W \subset Y$ is $f^{-1}(V) \cap g^{-1}(W)$, which is open in X as the intersection of open sets;

(d) by the above point we have that the set $\{x \in X : f(x) = g(x)\} \subset X$ is closed, so if it contains a dense subset it must be the whole X;

(e) consider the map $\phi : X \times Y \longrightarrow Y \times Y$, $\phi(x, y) = (f(x), y)$. The graph of f is just $\phi^{-1}(\Delta_Y)$, and we are done as in c).

(4) (a) The set $\mathbf{R} \setminus \{y\}$ is an open neighborhood of x not containing y. X is not Haudorff because every nonempty open set is dense (which is an even stronger condition): the complement of an open set is finite, and so cannot contain a nonempty open set;

(b) this graph is by definition just the diagonal Δ_X, which is not closed by 3b);

(c) if f is constant then it is continuous as the preimage of any point is either empty or the whole $\mathbf{R}$. Suppose f is not constant: then the preimage of any point is never the whole $\mathbf{R}$, and hence must be a finite set by continuity as the proper closed sets are precisely the finite sets (hence in particular points are closed). This precisely means that the equation $f(x) = y$ has only finitely many solutions for any $y \in \mathbf{R}$. Moreover, since the preimage of an union is the union of the preimages, the condition of having finite fiber over any point is also sufficient for continuity, as all closed sets are finite and hence the finite union of their points: in particular, bijections are continuous;

(d) just take f as the identity and g as the identity on $\mathbf{R} \setminus \{0, 1\}$ and $g(0) = 1, g(1) = 0$. They are bijections by construction but $f - g$ is identically 0 except at $0, 1$, so it is not constant but the equation $(f - g)(x) = 0$ has infinitely many solutions, and hence it is not continuous by c);

(e) in 3c), 3d) take $X = Y$ as our X and f, g as above. Then the second subset of 3c) is $U = \mathbf{R} \setminus \{0, 1\}$, which is not closed as it is proper and infinite, and the first one is $U \times U \cup \{0, 1\} \cup \{1, 0\}$ which again is proper in $\mathbf{R}^2$ (it does not contain $(0, 0)$) but not finite; since all closed sets in $X \times X$ are intersections of finite unions of product of closed sets, they are finite (as closed proper sets $C \subset X$ are), and hence $U \times U \cup \{0, 1\} \cup \{1, 0\}$ is not closed.
Finally, our f and g agree on the dense subset $X \setminus \{0, 1\}$ but are not equal, which is a "counterexample" to 3d).

(5) (a) Let $\{U_i\}_{i \in I}$ be a covering of $A_1 \cup A_2$; then $\{U_i \cap A_k\}_{i \in I}$ is a covering of A_k for $k = 1, 2$, so from each we can extract a finite subcovering, i.e. there are positive integers A, B and indeces $i_1, ..., i_A, j_1 ..., j_B$ such that $\{U_{i_1} \cap A_1, ..., U_{i_A} \cap A_1\}$ is a covering of A_1 and $\{U_{j_1} \cap A_2, ..., U_{j_B} \cap A_2\}$ is a covering of A_2. From this it follows that $\{U_{i_1}, ..., U_{i_A}, U_{j_1}, ..., U_{j_B}\}$ is a covering of $A_1 \cup A_2$, which proves it is a compact subset;

(b) we saw that if X is Hausdorff, a compact subset is closed, so $A_1 \cap A_2$ is closed as the intersection of two closed sets. But then it is closed also as a subspace of A_1 with the subspace topology, so it is a closed subset of a compact topological space, and hence itself compact.

(6) (a) We have $(x_0, y) \notin \Gamma_f$ with the latter being closed in $X \times Y$ in virtue of 3e), as Y is Hausdorff, so there is an open neighborhood $W \subset X \times Y$ of (x_0, y) such that $W \cap \Gamma_f = \emptyset$. By definition of the product topology, for any point $(x, y) \in W$, W contains the product of an open set of X containing x with one of Y containing y, so choosing $x = x_0$ (y was already arbitrary) we get the desired claim;

(b) $Y \setminus V$ is closed in a compact space and hence is compact. For any $y \in Y \setminus V$ we can find thanks to a) open sets $U_y \ni x_0$ and $V_y \ni y$ such that $(U_y \times V_y) \cap \Gamma_f = \emptyset$; by definition the V_ys form a covering of $Y \setminus V$ and we can therefore extract a finite subcovering $\{V_{y_i}\}_{i=1}^N$. But then we have that $U = \cap_{i=1}^N U_{y_i} \subset X$ is an open neighborhood of x_0 such that $U \times (Y \setminus v) \cap \Gamma_f = \emptyset$ by construction;

(c) f is continuous iff for any open set of Y its preimage is open, i.e. if for any (x_0, y_0) as in the hypotheses and *any* open neighborhood V of y_0 there is an open neighborhood U of x_0 that maps inside V. This is precisely what point b) says.

(7) The "only if" part follows from the definition of compactness. For the other direction, let $\mathscr{U} = (U_i)_{i \in I}$ be an open covering of X and let us pick for any $x \in X$, using the AC, an $i_x \in I$ such that $U_{i_x} \ni x$. Then we have a subcollection $(U_{i_x})_{x \in X}$ which is still a covering as any x is contained in its respective open set. By hypothesis there is a finite set $S \subset X$ such that $\mathscr{V} = (U_{i_x})_{x \in S}$ is still a covering, so indeed $\mathscr{U}$ admits a finite subcovering $\mathscr{V}$.