TOPOLOGY SPRING 2024
SOLUTIONS SERIE 5

(1) (a) We just need to show that its complement is open, i.e. that for any $y \neq x$ there is a neighborhood V of y that does not contain x; this is directly implied by the Hausdorff property (which is actually stronger);

(b) $\emptyset$ is open by definition, and $\widetilde{X} = X \cup \{\eta\}$ is open as X is open in itself; any union $\bigcup_{i \in I} U_i$ of open sets in $\widetilde{X}$ can be written, by discarding any empty U_is which do not contribute, as $\bigcup_{i \in I}(V_i \cup \{\eta\}) = \{\eta\} \cup \bigcup_{i \in I} V_i$ with the V_is open in X, and hence is open by definition; finally, any finite intersection of open sets in the new topology is either empty (if one of them is) or equal to $\{\eta\} \cup \bigcap_1^N V_i$ with the V_is open in X, and therefore open.

(c) if the claim were false, we would have a nonempty open set $U \subset \widetilde{X}$ not containing η, which contradicts the definition of the toplogy on $\widetilde{X}$.

(2) (a) for any point $y \in A$ we can take open neighborhoods U_y, V_y of x, y with empty intersection. $\{V_y\}_A$ is an open covering of A, which is compact as a closed subspace in a compact space, so there are $y_1, ..., y_N \in A$ such that $V = \bigcup_1^N V_{y_i} \supset A$. But then we have that $U = \bigcap_1^N U_{y_i}$ is an open neighborhood of x contained in the closed subset $X \setminus V$, and hence $\overline{U} \subset X \setminus V$ is disjoint from A;

(b) as the base step is done ($U = U_0$), let us take care of the inductive step for index n : as C_n has empty interior there is $x \in U_{n-1}$ such that $x \notin C_n$ (otherwise $U_{n-1} \subset C_n^\circ$). Apply point a) with $A = C_n$ and such x to get an open neighborhood U' of x whose closure is disjoint from C_n. Finally, either $\overline{U'} \subset U_{n-1}$ and we are done by taking $U_n = U'$, or to get U_n, apply a) again for x and $A = X \setminus U_{n-1}$ (which is nonempty as U_{n-1} does not contain $\overline{U}$), getting another open neighborhood V, and set $U_n = U \cap V$, which then satisfies the second condition (and also the first as it is a subset of U');

(c) if the intersection were empty, then $\{X \setminus \overline{U_n}\}_{n \geq 1}$ would be an open covering of X, thus admitting a finite subcovering, and hence we would get an N such that $\bigcap_1^N \overline{U_n} = \emptyset$, which is absurd as the second condition implies that this intersection is just $\overline{U_N}$, which is nonempty as the closure of a neighborhood of some x; finally, if $U \cap (X \setminus C) = \emptyset$, we would have $U \subset C$. But taking $x \in \bigcap_1^\infty \overline{U_n} \subset U$, we have $x \in \overline{U_n}$ $\forall n$ by definition and hence $x \notin C_n$ $\forall n$ by the first condition, yielding $x \notin C$;

(d) the previous point is exactly saying that U is not contained in C; as U is an arbitrary nonempty open set, we get that C has empty interior;

(e) the sequence $(C_n)_{n\geq 1}$, $C_n = X \setminus V_n$ satisfies the hypotheses above in virtue of the density condition, so its union has empty interior, which means that the complement of $V = \bigcap_1^\infty V_i$ has empty interior, and hence V is dense;

(f) set $V_n = X \setminus R_n$ where R_n consists of the (finitely many) rational numbers in $[0,1]$ which, when written in minimal terms, have denominator $\leq n$. So clearly V_n is dense and open. But the intersection is the irrational numbers in $[0,1]$, which is not open as $\mathbf{Q}$ is dense in $\mathbf{R}$.

(3) (a) The empty set and Y have the empty set and X, which are open, as preimages. The other two properties follow immediately from the fact that the union of the preimages of a collection of sets is the preimage of their union, and the same for the intersection;

(b) injectivity is clear, as no two points of X with different $\mathbf{R}$-coordinate are identified. For continuity, let $U \subset Y$ be open and nonempty: then $p^{-1}(U) \neq \emptyset$ is open in X; we distinguish three cases:
- if $|U \cap \{o_+, o_-\}| \neq 1$, then $p^{-1}(U)$ is of the form $V \times \{\pm 1\}$ with $V \subset \mathbf{R}$ open and $i_\epsilon^{-1}(U) = V$, so we are done;
- if $U \cap \{o_+, o_-\} = o_\epsilon$, $p^{-1}(U)$ is of the form $(V \setminus \{0\}) \times \{\pm 1\} \cup (0, \epsilon)$ with $V \subset \mathbf{R}$ open containing 0, and we have $i_\epsilon^{-1}(U) = V$, so we are done;
- if $U \cap \{o_+, o_-\} = o_{-\epsilon}$, $p^{-1}(U)$ is of the form $(V \setminus \{0\}) \times \{\pm 1\} \cup (0, -\epsilon)$ with $V \subset \mathbf{R}$ open containing 0, and we have $i_\epsilon^{-1}(U) = V \setminus \{0\}$ which is still open, so we are done;

(c) $Im(i_+)$ consists of the classes of all elements of the form $(x, 1)$, i.e. of $Y \setminus [(0,-1)] = Y \setminus o_-$ by the identification. As i_+ is continuous and injective, to prove it is a homeo on the image we just need to prove that the inverse is continuous: given $V \subset \mathbf{R}$ open, we need to prove that $p^{-1}(i_+(V)) \subset X$ is open. But if $0 \notin V$ this set is just $V \times \{\pm 1\}$, otherwise it is $V \times \{1\} \cup (V \setminus \{0\}) \times \{-1\}$, which is still open;

(d) by the homeomorphism of the previous point, it suffices to show that for each y there is an $\epsilon \in \{\pm 1\}$ and a neighborhood U of y with $U \subset Y \setminus \{o_\epsilon\}$. But this is clear, letting ϵ be any $\{\pm 1\}$-coordinate appearing in the preimage of y, the neighborhood $U = i_\epsilon(V)$ with V any $\mathbf{R}$-neighborhood of $i_\epsilon^{-1}(y)$ works;

(e) it suffices to push forward via i_ϵ as above a countable fundamental system V_n for $i_\epsilon^{-1}(y)$;

(f) for any sequence $(x_n)_{n\geq 1}$ of nonzero reals with $\lim_{n\to\infty} x_n = 0$, the sequence $([(x_n, 1)])_{n\geq 1}$ in Y converges to both origins (as it is equal to the sequence $([(x_n, -1)])_{n\geq 1}$).

(4) (a) Sum and product of functions are defined pointwise (and are clearly still functions $X \longrightarrow \mathbf{C}$), and $(\mathbf{C}, +, \cdot)$ is a commutative ring with multiplicative identity 1, so $\mathscr{C}(X)$ is indeed a ring with the specified operations;

(b) we have $|f(x)|^2 = f(x)\overline{f(x)} \in I$ as $f \in I$ and $\bar{f} \in \mathscr{C}(X)$;

(c) since the operations are defined pointwise, this amounts to saying that $0+0 = 0$ and $0z = 0$ $\forall z \in \mathbf{C}$;

(d) let $g \in \mathscr{C}(X)$ be arbitrary. Then $g = f \times \frac{g}{f}$ and $\frac{g}{f} \in \mathscr{C}(X)$ as it is well-defined since $f(x) \neq 0$ $\forall x \in X$, so $g \in I \Longrightarrow I = \mathscr{C}(X)$;

(e) the hypothesis tells us that for every $x \in X$ there is $f_x \in I$ such that $f_x(x) \neq 0$; since f_x is continuous and $\mathbf{C}$ is Hausdorff, we can find a neighborhood U_x of x such that $f_x(y) \in V_x$ $\forall y \in U_x$ with V_x a neighborhood of $f_x(x)$ not containing 0, so we are done;

(f) by d) we just need to construct a function $f \in I$ which is never 0. For all $x \in X$ take f_x as above: then the U_x's are an open covering of X, so there is a finite subcovering $(U_i)_1^N$. But then we have by b) that $|f_i|^2 \in I, i = 1, ..., N$, and hence $x \to \sum_1^N |f_i(x)|^2$ is in I, and it never vanishes, as by the previous point and the covering condition, for any $y \in X$ there is $1 \leq i \leq N$ such that $f_i(y) \neq 0 \Longrightarrow |f_i(y)|^2 > 0$ (and all other summands are nonnegative);

(g) we now know that if I is not contained in m_{x_0} for some x_0 then it is the whole $\mathscr{C}(X)$; so if I is maximal in particular it is contained in some m_{x_0}, which is an ideal by c), and hence I must be equal to that m_{x_0} by maximality. Conversely, m_{x_0} is an ideal, and if it were contained in another proper ideal J this would have to be (contained in) some other m_{x_1}, but clearly we never have a containement of the form $m_{x_0} \subset m_{x_1}$ for $x_0 \neq x_1$, as the indicator function of $\{x_1\}$ is in m_{x_0}, so m_{x_0} is maximal.