TOPOLOGY SPRING 2024
SOLUTIONS SERIE 6

(1) (a) If $\mathscr{F}$ is principal then by definition there is $x \in X$ such that $\mathscr{F} = \{S \subset X : x \in S\}$, so in particular $\{x\} \in \mathscr{F}$. If $\mathscr{F}$ contains a finite set $A = \{x_1, ..., x_n\}$ then we can show by induction on n that it contains a singleton: if $n = 1$ we are done, otherwise we know that $\mathscr{F}$ contains either $\{x_n\}$ or $X \setminus \{x_n\}$; in the former case we are done, and otherwise $\mathscr{F}$ contains $A \cap (X \setminus \{x_n\}) = \{x_1, ..., x_{n-1}\}$ and we are done by the inductive hypothesis;

(b) the complements of finite sets are a filter F_0, as taking finite unions/subsets of finite sets gives finite sets. So we know that there is an ultrafilter $F \supset F_0$. But F cannot be principal as it would contain a finite set by the previous point, and hence it would not contain its complement, which is absurd;

(c) proceeding as in the hint, we get that $X \setminus A \in \mathscr{F}$ and hence $\mathscr{F} \ni (A \cup B) \cap (X \setminus A) = B \setminus A$. As this is a subset of B, it follows that $B \in \mathscr{F}$;

(d) As $A \cap B$ is empty, they cannot both belong to $\mathscr{F}$, as otherwise $X \setminus A \supset B$ would belong to $\mathscr{F}$, contradicting the ultrafilter condition; by the previous point we know that $A \cup B \in \mathscr{F} \iff$ one of A, B does, so we are done;

(e) we have $\nu(A) = \nu(A \setminus B) + \nu(A \cap B)$ and similarly for B, so we get

$$\begin{aligned}\nu(A) + \nu(B) - \nu(A \cap B) &= \nu(A \cap B) + \nu(A \setminus B) + \nu(B \setminus A) = \\ &= \nu(A \cap B) + \nu(A \triangle B) = \nu(A \cup B),\end{aligned}$$

where $A \triangle B$ is the symmetric difference;

(f) if $A \subset B$ and $\nu(A) = 1$ then $\nu(B) = \nu(A) + \nu(B \setminus A) \geq \nu(A)$, but also $\nu(B) \in \{0, 1\}$ and hence $\nu(B) = 1$; if $\nu(A) = \nu(B) = 1$ then $\nu(A \cap B) = \nu(A) + \nu(B) - \nu(A \cup B) = 1 + 1 - 1 = 1$ by the previous point, and finally $\nu(\emptyset) = 0$ trivially by the hypothesis, so $\mathscr{F}_\nu$ is a filter. By the hypothesis we also get $1 = \nu(X) = \nu(A) + \nu(X \setminus A)$ $\forall A \subset X$, so it is also an ultrafilter.

(2) (a) If B was disconnected we would have open sets U_1, U_2 with $B \subset U_1 \cup U_2$ and $B \cap U_i$ nonempty and disjoint from each other for $i = 1, 2$. But then also $A \subset U_1 \cup U_2$ and the $A \cap U_i$ would be disjoint as $A \subset B$; finally we could not have $A \subset U_i$ for any i, as otherwise $X \setminus U_j, j \neq i$, would be a closed set containing A but not B, contradicting the fact that $B \subset \overline{A}$: hence, $A = (A \cap U_1) \sqcup (A \cap U_2)$ is disconnected, absurd;

(b) A is connected as the image of the the connected interval $(0, \infty)$ under the continuous map $f : \mathbf{R} \longrightarrow \mathbf{R}^2,\ x \mapsto (x, \sin(1/x))$;

(c) let S be the segment we added; the claim follows from point a) as $B \subset \overline{A}$ (they are in fact equal): for any $P \in S$ and any $\epsilon > 0$, $B_{\mathbf{R}^2}(P, \epsilon) \cap A \neq \emptyset$ as we can find $x > (\epsilon)^{-1}$ such that $\sin(x) = y_P$ by the continuity and periodicity of the sine over $\mathbf{R}$ and the fact that $-1 \leq y_P \leq 1$.

(3) (a) The complement of an intersection is the union of the complements, so we get the desired claim, since the equality $X \setminus \overline{A} = (X \setminus A)^\circ$ (and analogously $X \setminus \overline{X \setminus A} = A^\circ$) is true as both containements follow by definition;

(b) if $B \setminus \partial A = \emptyset$, then $B \subset A^\circ \sqcup (X \setminus A)^\circ$ by the previous point, and the claim follows by the connectedness of B;

(c) this follows immediately from the previous point;

(d) as X is connected we can apply the result of the above point for $B = X$, obtaining that $\partial A = B \cap \partial A \neq \emptyset$;

(e) writing $X = A \sqcup B$ with A, B open and nonempty, we have $\overline{A} = A, \overline{B} = B$ and hence $\partial A = A \cap B = \emptyset$ by a).

(4) If A is nonempty and not a singleton, it contains distinct elements $x = (x_k)_{k\geq 1} \neq y = (y_k)_{k\geq 1}$. So there is an n such that $x_n \neq y_n$. But then $U = \{z \in C : z_n = x_n\}$ and $V = \{z \in C : z_n = y_n\}$ are disjoint open sets (they are an open neighborhood of any of their points by definition of the topology on C) such that $U \sqcup V = C \supset A$. Since both $U \cap A$ and $V \cap A$ are nonempty, we are done. (This proof can be given in the context of the hint: the p_ks are continuous by definition of the topology, but p_n is not constant on A for n the same index as above).

(5) (a) $\mathbf{R}^0$ is a point and $\mathbf{R}^1$ is an interval in $\mathbf{R}$, and we know these are connected. We proceed by induction on d: consider the map $\rho : \mathbf{R} \times \mathbf{R}^{d-1} \longrightarrow \mathbf{R}^d,\ (x, v) \mapsto (x, v_1, ..., v_{d-1})$, which is clearly continuous (it is an identification map), and define $\rho_v : \mathbf{R} \longrightarrow \mathbf{R}^d,\ x \mapsto \rho(x, v)$ and $\rho_x : \mathbf{R}^{d-1} \longrightarrow \mathbf{R}^d,\ v \mapsto \rho(x, v)$ which are also continuous as precompositions of ρ with continuous immersions. So given continuous $f : \mathbf{R}^d \longrightarrow \{0, 1\}$, we have that $f \circ \rho_v$ is continuous from $\mathbf{R}$ to $\{0, 1\}$ and hence is constant; if we define $U_0 = \{v \in \mathbf{R}^{d-1} : f \circ \rho_v(x) = 0\ \forall x \in \mathbf{R}\}$ and $U_1 = \{v \in \mathbf{R}^{d-1} : f \circ \rho_v(x) = 1\ \forall x \in \mathbf{R}\}$ we then have $U_0 \sqcup U_1 = \mathbf{R}^{d-1}$ but also $U_i = (f \circ \rho_x)^{-1}(i)$, so they are open, and since $\mathbf{R}^{d-1}$ is connected by the inductive hypothesis, one of them is empty, so f is constant;

(b) $\mathbf{R}\setminus\{0\} = (-\infty, 0) \cup (0, \infty)$ is the disjoint union of two nonempty open subsets, and therfore disconnected. As these two are homeomorphic to intervals of $\mathbf{R}$ (via the identity), they are connected, and in particular the two connected components;

(c) the hyperspherical coordinates give an homeomorphism $\mathbf{R}^{d-1}\setminus\{0\} \simeq (0, \infty) \times S^1 \times [0, \pi]^{d-2}$. Since S^1 and $(0, \infty)$ are connected, the proof of a) (which in fact shows that the product of connected spaces is connected) directly adapts for $d = 2$. Then, for $d > 2$, we can proceed analogously by induction on the exponent of $[0, \pi]$;

(d) if $d = 0$, $S^d = \{-1, 1\}$ is disconnected as a discrete set with more than one element. For $d \geq 1$, the projections $\mathbf{R}^{d+1} \setminus \{0\} \longrightarrow S^d$, which in the coordinates of the previous point are given by $(r, \phi, \theta_1, ..., \theta_{d-1}) \mapsto (1, \phi, \theta_1, ..., \theta_{d-1})$ are clearly continuous and surjective, so S^d is also connected (otherwise, the preimages of two disjoint open subsets disconnecting S^d would disconnect $\mathbf{R}^{d+1}$);

(e) the strategy of the hint works because of a lemma seen in class. As C_s we can take the union of the sphere of radius s centred at the origin with the segment connecting the origin with $(r, 0, ..., 0)$. So clearly these have nonempty intersection given by this segment, and their union is B_r. But they are also connected, as we can again apply the lemma to the union that defines them, since both spheres and segments are connected, and they intersect in $(s, 0, ..., 0)$, so we are done.

(6) (a) Let $P_0 = (0, 1)$ be the missing point. The stereographical projection from P_0 $\pi : X \longrightarrow \mathbf{R}$, $P \mapsto l_P \cap \{y = 0\}$ where l_P is the line connecting P and P_0 is a well-known homeomorphism (all the verifications are trivial), and since $\mathbf{R}$ is connected, X is;

(b) we can just take $B = B((1, 0), \sqrt{2} + \epsilon)$ for any $0 < \epsilon < 2 - \sqrt{2}$; indeed, we then have $(-1, 0) \notin B \Longrightarrow B \subset Q \sqcup (X \setminus (Q \cup \{(-1, 0)\}))$, where $Q = \{\frac{\pi}{2} < \theta < \pi\}$ is the portion of S^1 contained in the **open** top left quadrant. Since $X \setminus (Q \cup \{(-1, 0)\}) = \{-\pi < \theta < \frac{\pi}{2}\}$ is open for the same reason, and both it and Q intersect B as $\epsilon > 0$, B is not connected.

(7) (a) Clearly if $a \leq s \leq t$ then $s \in G$ by definition of G, so we just need to show that if $t \in G$, then G contains a right neighborhood of t. If $t \in G$, then $[a, t] \subset \bigcup_{j \in J} U_j$, so there is j such that $t \in U_j$, and hence, since U_j is open, there is $\epsilon > 0$ such that $[a, b] \cap [t, t + \epsilon) \subset U_j$, which implies that any s in this right neighborhood is in G;

(b) consider the complement H of G in $[a, b]$: let us prove that H is open. Given $s \in H$, clearly $r \in H$ for all $r > s$; hence, we just need to prove that H contains a left neighborhood of s: as there is some $l \in I$ such that $s \in U_l$, there is $\epsilon > 0$ such that $(s - \epsilon, s] \subset U_l$, so if we could find some $r \in (s - \epsilon, s) \cap G$, we'd have $[a, s] \subset U_l \cup \bigcup_J U_j$ where $[a, r] \subset \bigcup_J U_j$ with J finite, contradicting the fact that $s \notin G$. So H is open and hence G is closed;

(c) as $[a, b]$ is connected, G is either empty or $[a, b]$. But clearly $a \in G$, so $G = [a, b]$ and hence $[a, b]$ is compact by definition of G.