# TOPOLOGY SPRING 2024
# SOLUTIONS SERIE 7

(1) (a) Since $d(x,y) = |x-y|$ is a distance on $X$, any function $D : X \times X \longrightarrow [0,\infty)$ of the form $D(x,y) = |f(x) - f(y)|$ is symmetric, nonnegative and satisfies the triangle inequality. In the case of our $\delta$, $f(x) = e^{-x}$ is injective, so $\delta(x,y) = 0 \iff x = y$, and hence it is a distance;

(b) clearly $f$ is bijective. Moreover, the open ball $B_\delta(x_0, r) = \{x \geq 0 : |f(x) - f(x_0)| < r\}$ is precisely the preimage under $f$ of $B_{\text{euc}}(f(x_0), r)$, which implies continuity of $f$ and its inverse (it is enough to verify it on a basis). So, the topology induced by $\delta$ is precisely the preimage under $f$ of the euclidean topology on $(0,1]$; as the exponential function is continuous on $\mathbf{R}$ with the euclidean topology, any restriction of it is, so any open set for $\tau_\delta$ is open in the euclidean topology. The converse follows from the same argument with the logarithm, so we are done;

(c) the sequence $x_n = n$ is Cauchy as, for $n \leq m$, $|e^{-n} - e^{-m}| < e^{-n}$ goes to 0 in $n$, but it does not converge in $X$, so $(X,\delta)$ is not complete

(d) we know $\mathbf{R}$ with the euclidean distance is complete (we will always take $\mathbf{R}$ with this distance in this point, without specifying it); let $i : X \hookrightarrow \mathbf{R}$ be the inclusion. Given a Cauchy sequence $(x_n)_{n\geq 0}$ in $X$ with the restriction of the euclidean distance, the sequence $(i(x_n))_{n\geq 0}$ has a limit $x \in \mathbf{R}$: we want to prove that $x \in i(X)$. The key idea is that $(i(x_n))_{n\geq 0}$ has to be contained in a compact set: otherwise, by the classification of compact subsets of $\mathbf{R}$, there are arbitrarily large elements in the sequence, which therefore cannot possibly have finite limit, and hence limit in $\mathbf{R}$. But then, $\exists C \subset [0,\infty)$ compact such that $i(x_n) \in C$ $\forall n \geq 0$ and hence $(i(x_n))_{n\geq 0}$ admits a subsequence converging in $C$, but since the whole sequence converges the limit must be the same, so $x \in C \subset i(X)$ and we are done.

(2) (a) Take a sequence as in the hint. By the inclusion and diameter hypotheses, for $n \leq m$ we have $d(x_n, x_m) \leq \text{diam}(C_n) \to 0$ as $n \to \infty$, so the sequence is Cauchy and hence has a limit $x \in X$. Let $C$ be the intersection and suppose $x \notin C$: then there is $N$ such that $x \notin C_N$. Since the $C_n$'s are closed, this means that there is a neighborhood $B_d(x,\epsilon)$ of $x$ disjoint from $C_N$, and hence from any $C_n$ with $n \geq N$; but this implies that $\inf_{y\in C_n} d(x,y) \geq \epsilon$ $\forall n \geq N$, and since $x_n \in C_n$ the sequence cannot converge to $x$, which gives the desired absurd;

(b) we can just take $(X,d) = (X,\delta)$ from Exercise 1, $C_n = [n,\infty)$ which are closed by 1b) and $x_n = n \in C_n$ which has no limit in $X$. Since the $C_n$'s satisfy the containment condition and have diameter $e^{-n} \to 0$ as $n \to \infty$, this is a counterexample for a non-complete $(X,d)$.

(3) (a) As with the topological version of Baire's Theorem, we proceed by induction. To construct $U_n, n \geq 1$ we take $x \in U_{n-1} \setminus C_n$ (which exists by the empty interior condition) and consider the sequence $(V_k)_{k\geq 1}$ of open sets given by $V_k = B(x, \frac{1}{k})$: we claim that there is $k$ such that $V_k$ satisfies the three required properties. Clearly the last one is satisfied for $k \geq 2n$; for the first two we just need to find $k$ such that $\overline{V_k} \cap Y_n = \emptyset$, where $Y_n$ is the closed set $C_n \cup (X \setminus U_{n-1})$. If this intersection was nonempty for all $k \geq 1$, we would have a sequence $(y_k)_{k\geq 1}$ of elements with $y_k \in \overline{V_k} \cap Y_n$, and since $Y_n$ is complete by what you have seen in class, this converges to some $y \in Y_n$, which therefore satisfies $\lim_{k\to\infty} d(y_k, y) = 0$. On the other hand, we have $y_k \in \overline{V_k}$ and $\bigcap_{k\geq 1} \overline{V_k} = \{x\}$ as a metric space is Hausdorff, so $\lim_{k\to\infty} d(y_k, y) = d(x, y) \neq 0$, absurd. So, choosing $U_n = V_k$, we are done;

(b) this follows from 2a) applied to $C_n = \overline{U_n}$ (not to be confused with this Exercise's $C_n$'s). Indeed these satisfy the containment hypothesis by the second condition and the diameter hypothesis by the third condition;

(c) let $x$ be a point in the above intersection: then $x \in U$ by the second condition, so if we had $U \subset C$ we would have $x \in C \implies \exists N : x \in C_N$. But since by definition $x \in \overline{U_n}\ \forall n \geq 1$ we have $x \notin C_n\ \forall n \geq 1$ by the first condition, which gives the desired absurd.

(4) (a) Let $n \geq m \geq N > 0$. Then the hypothesis and the triangle inequality give

$$d(x_n, x_m) = d(f^{(m)}(x_{n-m}), f^{(m)}(x_0)) \leq \alpha^m d(x_{n-m}, x_0) \leq \alpha^N \sum_{k=0}^{n-m-1} d(x_{k+1}, x_k),$$

as $\alpha \leq 1$, where $f^{(k)}$ denotes the $k$-th iterate of $f$. But $d(x_k, x_{k-1}) \leq \alpha^k d(x_1, x_0)$ again by the hypothesis, so we get $d(x_n, x_m) \leq \frac{\alpha^N}{1-\alpha} d(f(x_0), x_0)$ by the geometric series formula. This goes to 0 independently of $x_0$ and $\alpha < 1$ as $N$ goes to $\infty$, so the sequence is Cauchy, and hence converges to some $y \in X$ by completeness.

Now we want to prove $f(y) = y$; suppose it is not: then there is an open neighborhood $V$ of $f(y)$ not containing $y$. By continuity of $f$, $U = f^{-1}(V)$ is an open neighborhood of $y$, so it contains a tail $(x_n)_{n\geq N}$ of the sequence. But then we have that the subtail $(x_n)_{n>N} = (f(x_n))_{n\geq N}$ lies in $V$, and therefore cannot converge to $y$, which is absurd;

(b) if $z$ satisfies the fixed-point functional equation, we have $d(y, z) = d(f(y), f(z)) \leq \alpha d(y, z)$, which for $\alpha < 1$ implies $d(y, z) = 0$, i.e. $y = z$.

(5) (a) $X$ is a length 1 "comb" with infinitely many vertical "teeth" of length 2, in particular one attached at each extreme $\{0, 1\}$ of the horizontal segment and the others accumulating to the 0 extreme as $\{1/n\}$. Let $H$ be the horizontal segment and $T_n,\ n > 0$ be the tooth attached at $1/n$, while we denote $T_0$ the tooth attached at 0. To show $X$ is connected we resort to the usual lemma seen in class, writing $X = \bigcup_{\mathbf{N}} A_n$ where $A_n = H \cup T_n$ for $n \geq 0$. Then $\bigcap_{\mathbf{N}} A_n = H \neq \emptyset$, and each $A_n$ is clearly connected: $A_0$ and $A_1$ are homeomorphic to an interval of $\mathbf{R}$ (by straightening the right angle) and

for the other $A_n$'s we can simply reapply the lemma to the decomposition $A_n = (T_n \cup (H \cap \{x \leq 1/n\})) \cup (T_n \cup (H \cap \{x \geq 1/n\}))$. Therefore, $X$ is connected;

(b) they do because $X$ has the subspace euclidean topology and the $U_\delta$ are the intersection of $X$ with a fundamental system of open neighborhoods of $(0,0)$ in $\mathbf{R}^2$;

(c) for such $\delta$, $U_\delta$ is disjoint from $H$, so it consists of the disjoint union of infinitely many vertical open segments of length $2\delta$; the connected component of $x_0$ in $U_\delta$ is therefore the segment containing it, i.e. $\{0\} \times (-\delta, \delta)$;

(d) given any fundamental system $(V_i)_I$ of open neighborhoods of $x_0$, for any $\delta > 0$ there must be $i = i(\delta)$ such that $U_\delta \supset V_i$, so the above argument gives that the connected component of $x_0$ in $V_i$ is a subset of $\{0\} \times (-\delta, \delta)$, and hence $V_i$ is not connected as $V_i$ must itself contain some $U_{\delta'}$, $0 < \delta' < 1$, and hence infinitely many vertical open segments. Since $X$ has a point not admitting a fundamental system of connected open neighborhoods, it is not locally connected;

(e) any discrete subset of $\mathbf{R}$ with more than one element, like $\{0, 1\}$, clearly works.

(6) (a) The empty set is open in $X$ and also compact, so we just need to verify the union and finite intersection conditions: given a collection $\{U_i\}_{i \in I}$ of sets in $\mathscr{T}_\infty$, either all of them are subsets of $X$, in which case the topology conditions are satisfied by definition, or we have

$$\bigcup_I U_i = U \cup \{\infty\} \cup \bigcup_J (X \setminus C_j)$$

for an open set $U$ of $X$, nonempty $J$ and compact subsets $C_j,\ j \in J$ of $X$. Therefore, letting $C$ be the closed subset $X \setminus U$ of $X$, we have

$$\bigcup_I U_i = \{\infty\} \cup X \setminus (C \cap \bigcap_J C_j).$$

But as each $C_j$ is compact and $X$ is Hausdorff, each $C_j$ is closed, so fixing any $j_0 \in J$ we have that $C \cap \bigcap_J C_j = C_{j_0} \cap (C \cap \bigcap_{J \setminus \{j_0\}} C_j)$ is closed in a compact set and therefore compact, so the union still belongs to $\mathscr{T}_\infty$.
Letting $I$ be finite this time, we have that either $\infty \notin \bigcap_I U_i$, in which case the intersection is a finite intersection of sets in the topology of $X$ with finitely many sets of the form $X \setminus C$ with $C$ compact, and hence closed, so again open sets, and hence still belongs to $\mathscr{T}_\infty$, or if $\infty$ belongs to the intersection, we have

$$\bigcap_I U_i = \{\infty\} \cup (X \setminus \bigcup_I C_i),$$

with the finite union being of compact subsets of $X$, and hence itself compact by Exercise 5 of Serie 4;

(b) separating $x, y \in X$ can be done by restricting to the topology on $X$ as it is Hausdorff, so let WLOG $y = \infty$; we want to prove that there are an open neighborhood $U$ of $x$ and a compact set $C \subset X \setminus U$. Taking any such $U \neq X$ (which exists by the Hausdorff condition) and any $z \in X \setminus U$, we know that $z$ has a compact neighborhood $D$. So $D \cap (X \setminus U)$ is closed in a compact set, and hence compact, but also nonempty and disjoint from $x$, so we are done;

(c) let $U$ be open in $\hat{X}$; clearly if $U$ is an open subset of $X$ it is stable under $i$, so in particular $i^{-1}$ is continuous by the definition of the topology on $\hat{X}$. Otherwise, $U = \{\infty\} \cup (X \setminus C)$ and so $i^{-1}(U) = X \setminus C$ is the complement of a compact, and hence closed ($X$ is Hausdorff), subset of $X$, and hence open, so $i$ is continuous. Moreover, being an inclusion, $i$ is injective and surjective, so we are done;

(d) given an open cover $\{U_i\}_I$ of $\hat{X}$, it must contain an open set $U \ni \infty$, so $C = \hat{X} \setminus U$ is compact by definition of the topology; but the restriction $\{U_i \cap C\}_I$ is an open cover of $C$, so admits a finite subcover $\{U_n \cap C\}_1^N$, and hence $\{U\} \cup \{U_n\}_1^N$ is a finite open subcover for $\hat{X}$;

(e) only in point b). If $X$ is compact then $\{\infty\}$ itself is open, so we are just adding a discrete point.