TOPOLOGY SPRING 2024
SOLUTIONS SERIE 8

(1) (a) Using the same notation for the functions f_n as in the example of the lecture referenced in the hint, we can just take the sequence $\{f + \epsilon f_n\}_{n\geq 1} \subset B(f, \epsilon)$, which is again an infinite discrete closed subset as $d(f + \epsilon f_m, f + \epsilon f_n) = d(\epsilon f_m, \epsilon f_n) = 2\epsilon$, so the ball cannot be compact;

(b) if there was an f with such neighborhood U, U would contain an open ball $B(f, 2\epsilon)^\circ = \{g \in X : d(f, g) < 2\epsilon\}$ and hence the closed ball $B(f, \epsilon)$, which would then be compact as a closed subset of a compact set, hence an absurd by the previous point.

(2) (a) Given $x = (x_i) \neq y = (y_i)$, there must exists $j \in I$ such that $x_j \neq y_j$, so we can take $U_j, V_j \subset X_j$ disjoint open neighborhoods of x_j, y_j respectively. Then $U = \prod_i U_i, V = \prod_i V_i$ with $U_i = V_i = X_i$ for $i \neq j$ are disjoint open neighborhoods of x, y respectively;

(b) since a basis $\mathscr{B}$ for the product topology on X is given by the sets of the form $\prod_i U_i$ with $U_i \subset X_i$ open and $U_i = X_i$ for all but finitely many i, a basis for the subspace topology on Y is the intersection of the sets in $\mathscr{B}$ with Y, so subsets of the form $\prod_i V_i$ with $V_i \subset Y_i$ open (by definition of the subspace topology) and $V_i = Y_i$ for all but finitely many i; a basis for the product topology on Y is by definition the same, so we are done;

(c) let Y be as in the previous point and let $\pi_i : X \longrightarrow X_i$ be the natural projection; the largest open that doesn't intersect Y is $\bar{Y}^c = \bigcup_i \pi_i^{-1}((Y_i^c)^\circ)$, as the subspace and product topologies on Y are the same. But then its complement is $\prod_i \pi_i^{-1}(\bar{Y}_i) = \prod_i \bar{Y}_i$ (this is purely set theoretical), so the two subsets of the claim have the same complement and hence are equal;

(d) by the previous point we know $\overline{\prod_i C_i} = \prod_i \bar{C}_i = \prod_i C_i$, so we are done;

(e) it suffices to take any infinite I, $X_i = \mathbf{R}\ \forall i$ and $U_i = \mathbf{R}^+\ \forall i$. Then the product U is not open as no element of the basis used above is contained in U;

(f) the convergence of the sequence means that for each open neighborhood U of x there is n such that $x_n \in U$. So, if for each $i \in I$ we take a fundamental system of open neighborhoods $\{U_f^i\}_{f\in F}$ of x_i in X_i and apply this for $U = \pi_i^{-1}(U_f^i)$, we precisely obtain that $(x_{n,i})$ has to converge to x_i as n tends to ∞. This is also sufficient as we took fundamental systems of neighborhoods and so by definition any U is a union of finite intersections of the $\pi_i^{-1}(U_f^i)$'s;

(g) let C be the connected component of x and C_i be the connected component of x_i in X_i; we proved that the product of connected sets is connected in the previous Solutions sheet, so $\prod_i C_i \subset C$. To finish, we just need to prove that if $\pi_i(C) \subset C_i\ \forall i$. But $\pi_i(C)$ is the image under a continuous map of a connected space and hence is connected; as it contains x, it is contained in C_i by definition.

(3) (a) As $d(x,y) = \sum_{n\geq 1} 2^{-n} a_n$ with $a_n < 1$, we have that $d(x,y) < \sum_{n\geq 1} 2^{-n} = 1$. Clearly d is nonnegative and symmetric as a sum of nonnegative and symmetric quantities, and it is 0 when all summands are 0, so when $d_n(x,y) = 0$ for all n, so $x = y$. To prove the triangle inequality we note that the functions $a_n = a_n(x,y)$ as defined above are distances, as we proved in Exercise 1 of Serie 2, and hence satisfy the triangle inequality. But then, so does any linear combination of them (even infinite, as the inequality is true term-by-term);

(b) say $U \subset X$ is open in the product topology and fix $x = (x_n) \in U$. Then we can find a finite set $S \subset \mathbf{Z}^+$ and balls $B_{d_k}(x_k, \epsilon_k)$ for $k \in S$ such that $\bigcap_S p_k^{-1}(B_{d_k}(x_k,\epsilon_k)) \subset U$, where $p_k : X \longrightarrow X_k$ is the canonical projection. But then, setting $M = \min_S 2^{-k}\frac{\epsilon_k}{1+\epsilon_k}$ we immediately get $B_d(x,M) \subset \bigcap_S p_k^{-1}(B_{d_k}(x_k,\epsilon_k))$, so an open set for the product topology is open for d.
To prove the converse, take U and x as above and some open ball $B_d(x,\epsilon) \subset U$. Let $N = 1 + \lfloor -\log_2(\epsilon) \rfloor \geq 1$; as $\epsilon - 2^{-N} = \delta > 0$, setting $\alpha = \frac{\delta}{(1-\delta)N}$ (or $\alpha = \infty$ if $\delta = 1$) we get that

$$d(x,y) < \sum_{k=1}^{N} 2^{-k} \frac{\alpha}{(1+\alpha)N} + \sum_{n\geq N+1} 2^{-n} \leq \frac{\alpha}{1+\alpha} + 2^{-N} = \delta + 2^{-N} = \epsilon,$$

and therefore

$$\bigcap_{k=1}^{N} p_k^{-1}(B_{d_k}(x_k,\alpha)) \subset B_d(x,\epsilon) \subset U;$$

(c) let $(x^{(k)})_{k\in\mathbf{N}}$ be a Cauchy sequence in X; from the definition we immediately get that for any $\epsilon > 0$ there is $k = k(\epsilon)$ such that, for any $i,j > k$ and for any $n \geq 1$, $\frac{d_n(x_n^{(i)},x_n^{(j)})}{1+d_n(x_n^{(i)},x_n^{(j)})} < \epsilon 2^n$. This immediately implies that the sequence of n-th coordinates $(x_n^{(k)})_{k\in\mathbf{N}}$ in X_n is Cauchy as for any $\delta > 0$ we have $d_n(x_n^{(i)},x_n^{(j)}) < \delta$ for $i,j > k = k(\frac{\epsilon 2^n}{1-\epsilon 2^n})$ (if $\epsilon 2^n \geq 1$ we simply get $k = 0$). Hence it converges to some $x_n \in X_n$. But then $x = (x_n)_{n\geq 1}$ is an element of X, and so our original sequence converges to x since for any $\epsilon > 0$ we can find $K \in \mathbf{N}$ such that $d(x,x^{(K)}) < \epsilon$: just take as usual $N \geq 1$ such that $\sum_{n>N} 2^{-n} < \epsilon$, call δ their (positive) difference, and take K large enough that all the N distances $d_n(x_n,x_n^{(k)})$ for $n = 1,...,N$ are less than δ/N for $k \geq K$ (such K exists because of the coordinate wise-convergence and the fact that we are imposing the condition on **finitely many** coordinates);

(d) note the change of notation from the above point: now each superscript corresponds to a *sequence* of elements of X (whose index, along with the coordinates, are subscripts), and not to a single element.
Let us proceed by induction. $x^{(1)}$ is already defined (it does not strictly satisfy (3) as even if X_1 is compact we'd need to extract a subsequence, so say we do that); to extract $x^{(N)}$ from $x^{(N-1)}$ we need the additional condition that the

sequence of the N-th coordinates converges. As the sequence $(x_n^{(N-1)})_N$ (with n varying) lives in X_N which is compact, it admits a converging subsequence $(x_{n_i}^{(N-1)})_N$ with $i \in \mathbf{Z}^+$; now we can just define $x_j^{(N)} = x_{n_j}^{(N-1)}$ and we are done. Finally, the sequence $(x_N^{(N)})_{N\geq 1}$ is a (well-defined, by (2)) subsequence of (x_m) which converges by the same argument of c), since the sequences of its coordinates converge by (3);

(e) this follows directly from d) and the equivalence of definitions of compactness for metric spaces you saw in class.

(4) (a) We know that for any $x_2 \in X_2$ the maps $i_{x_2} : X_1 \longrightarrow X,\ x \mapsto (x, x_2)$ are homeomorphisms on the image, so since $f_{x_2} = f \circ i_{x_2}$, f_{x_2} is continuous as the composition of continuous maps. The same argument works for g_{x_1};

(b) since $\mathbf{R}^n$ is Hausdorff, continuous functions have a unique limit as the variable tends to a point, equal to the value of the function at that point. In our case, $f((0, 0)) = 0$ but the limit taken along the line $x = y$ is $\lim_{t\to 0} \frac{t^2}{t^2+t^2} = \frac{1}{2} \neq 0$, so f is not continuous.
On the other hand, if $x_2 \neq 0$ then $f_{x_2}(x) = \frac{xx_2}{x^2+x_2^2}$ is a formula that clearly defines f for *all* $x \in \mathbf{R}$, and is continuous as the product/composition of elementary continuous functions; if $x_2 = 0$ we get $f(x) \equiv 0$ which is also continuous. As f is symmetric, the claim also holds for g_{x_1};

(c) this is *precisely* the standard definition of continuity applied to g_{x_1} at the point $x = x_2$;

(d) we know $y - \epsilon < f(x_1, v_i) < y + \epsilon$ for $i = 1, 2$ by the previous point. Let $u_i = f(x_1, v_i)$ and let d_i be the smallest of the two distances $|u_i - (y \pm \epsilon)|$ of u_i from the extremes of the interval; then, by the continuity of f_{v_i} at $x = x_1$, we get that for each $\eta > 0$ there is $\delta > 0$ such that $u_i - \eta < f(x, v_i) < u_i + \eta$ for all $x_1 - \delta < x < x_1 + \delta$, for $i = 1, 2$. This, applied to any $\eta < \min_i(d_i)$ gives the claim;

(e) with the above notation, as (x_1, x_2) was arbitrary, the claim is equivalent to the following: for any $\epsilon > 0$ there exists $\mu > 0$ such that $y-\epsilon < f(x, t) < y+\epsilon$ for all (x, t) such that $x_1 - \mu < x < x_1 + \mu,\ x_2 - \mu < t < x_2 + \mu$. Take the two inequalities obtained in d) for $x \in (x_1 - \delta, x_1 + \delta)$; for any such x, by the monotonicity hypothesis, we have $y-\epsilon < f(x, v_1) \leq f(x, t) \leq f(x, v_2) < y+\epsilon$ for any $t \in (v_1, v_2)$: therefore, by taking $\mu = \min(\delta, x_2 - v_1, v_2 - x_2)$, we are done.