TOPOLOGY SPRING 2024
SOLUTIONS SERIE 9

(1) (a) The complex exponential map $z \mapsto \exp(2\pi i z)$ is continuous, 1-periodic and maps $\mathbf{R}$ to the unit circle $C(0,1)$ in $\mathbf{C}$, so the first branch of ϕ maps $[0,1]$ to $C(i,1)$ and the second branch maps $[1,2]$ to $C(-i,1)$ by the additive translations. We know both branches are continuous so we just need to check that ϕ is well-defined at the common point of the domains of both branches, $t=1$: for the first branch we have $\phi(1) = i + \exp(\frac{3}{2}\pi i) = 0$ and for the second we have $\phi(1) = -i + \exp(\frac{\pi i}{2}) = 0$;

(b) we construct $\tilde{\phi}$ as follows: let $c \in X'$ be the equivalence class $\{0,1,2\}$, so that $X' = c \cup \bigcup_{0<t<2, t\neq 1} \{t\}$. Then for the triangular diagram to commute we must have $\tilde{\phi}(\{t\} = \phi(t)$ and $\tilde{\phi}(c) = \phi(i)$ for $i = 0,1,2$. We know that $\phi(1) = 0$, so our $\tilde{\phi}$ is well-defined iff $\phi(0) = \phi(2) = 0$, which a check with the definition immediately verifies (note how we just proved that $\tilde{\phi}$ is, set-theoretically, unique).
To prove continuity, notice that the quotient map p is a homeomorphism when restricted to the open $U = (0,1)\cup(1,2)$, and that $0 \notin \phi(U)$ as the exponential function is injective on both branches defining ϕ. So given $V \subset Y$ open such that $c \notin \tilde{\phi}^{-1}(V)$, the latter satisfies $p^{-1}(\tilde{\phi}^{-1}(V)) = \phi^{-1}(V)$ and hence is open; if $c \in \tilde{\phi}^{-1}(V)$ then we have $p^{-1}(\tilde{\phi}^{-1}(V)) = \{0,1,2\} \cup p^{-1}(\tilde{\phi}^{-1}(V) \setminus \{c\})$, so we just need to prove that the latter contains sets of the form $(0,\epsilon)$, $(1-\epsilon, 1+\epsilon)$, $(2-\epsilon, 2)$. But then, as $0 \notin \phi(U)$, this follows from the fact that $p^{-1}(\tilde{\phi}^{-1}(V) \setminus \{c\}) = \phi^{-1}(V \setminus \{0\})$, which is open as $V \setminus \{0\}$ clearly is so in Y (we have $V = Y \cap D$ with some $D \subset \mathbf{C}$ open, so $V \setminus \{0\} = Y \cap (D \cap (\mathbf{C} - \{0\})))$, and must intersect any neighborhood of a point in $\phi^{-1}(0)$ (as otherwise we'd get such an open neighborhood N with $\phi(N) \cap V = \{0\}$; up to restricting N we can make it so only one of $\{0,1,2\}$ belongs to it, so that $\phi|_N$ is a homeomorphism and hence $\phi(N)$ is open, giving $\{0\} \subset Y$ open, which is clearly absurd);

(c) in the first point we proved $\tilde{\phi}$ is continuous, but also (along with the first observation of the second point) a bijection. Given $U' \subset X'$ open, we have $p^{-1}(U')$ open, so we just need to prove that ϕ is an open map as the triangle commutes. This is true as the exponential is a homeomorphism on each branch, and the image of a neighborhood of 1 is just the union of the left and right parts of it, which is clearly still open as a subset of Y.

(2) (a) First, we need to show that the map described in the exercise is well defined, that is, if (x,y) and (x',y') have the same class in Z' then $p_X(x) = p_X(x')$ and $p_Y(y) = p_Y(y')$. But two pairs have the same class in Z' exactly when the first coordinates are equivalent in X and the second ones are equivalent in Y,

which means precisely that they have the same projections in the respective quotients. Note how these are logical equivalences, so we also proved that ϕ is injective. Surjectivity is clear as the natural coordinate-wise map $(p_X, p_Y) : Z \longrightarrow X' \times Y'$ is surjective and ϕ makes the diagram with (p_X, p_Y) and p commute by definition.
For continuity, let V the preimage of an open set W under ϕ. Then, in virtue of the quotient topology, we just need to prove that $p^{-1}(V)$ is open, but by the diagram described above we have $p^{-1}(V) = (p_X, p_Y)^{-1}(W)$ which is open as the map is continuous;

(b) (note the change of notation from the previous point) as the union of the images is the image of the unions, we can clearly prove this for W's forming a basis of the product topology, so let $W = U \times V$ with $U \subset X, V \subset Y$ open. By the commutativity we already used above, we have $\phi(p(W)) = (p_X, p_Y)(W) = p_X(U) \times p_Y(V)$, which is open since p_X, p_Y are open;

(c) we already knew that ϕ was bijective and continuous, so we just need it to be open. In virtue of the quotient topology, any open set $T \subset Z'$ is of the form $p(W)$ for some $W \subset Z$ open (this is equivalent to saying that $p^{-1}(T)$ is open), so we are done by b).

(3) (a) We have that

$$q^{-1}(q(\Delta)) = \{(x, y) \in X \times X : \exists z \in X : x \sim z \ \& \ y \sim z\} = \{(x, y) \in X \times X : x \sim y\} = \Gamma$$

by transitivity of the equivalence relation;

(b) as $\Gamma = q^{-1}(q(\Delta))$ is closed this means that $q(\Delta) = q(\Gamma)$ is closed in $Y = (X \times X)/\equiv$ in virtue of the quotient topology; by Exercise 1 we know that there is a bijection $\phi : Y \longrightarrow X' \times X'$ sending the class of (x, y) to $(p(x), p(y))$, hence bijecting $q(\Gamma)$ to $\{(p(x), p(x)) : x \in X\}/\equiv\, = \Delta'$. Since p is open we have that $\phi(q(W))$ is open for any $W \subset X^2$ open, again by 1 b), so taking $W = \Gamma^c$ we get that $\Delta'^c = \phi(q(\Gamma))^c = \phi(q(W))$ is open, giving the desired claim;

(c) this is precisely point b) of the Exercise in the hint.

(4) Let us call $0_Y \in Y$ the class of $\mathbf{Z}$.

(a) p_X is the identity, so it's open;

(b) p_Y is the identity on the open $\mathbf{R} \setminus \mathbf{Z}$, which gives the first case; moreover, p_Y is constantly 0_Y over $\mathbf{Z}$, giving the second;

(c) for example we have $p_Y((-1/2, 1/2)) = (-1/2, 0) \cup (0, 1/2) \cup \{0_Y\}$ which is not open as its preimage $(-1/2, 1/2) \cup (\mathbf{Z} - \{0\})$ is not open in $\mathbf{R}$. To prove the map is closed we just need to prove that $p_Y^{-1}(p_Y(C))$ is closed for $C \subset \mathbf{R}$ closed, in virtue of the quotient topology, and this follows directly from b) as $\mathbf{Z} \subset \mathbf{R}$ is closed;

(d) clearly a fundamental system of neighborhoods (FSN) of a point in a (**finite-**)product space (why finite?) is given by the product of a FSN for each coordinate, so we just need to prove that $p_Y(\bigcup_{\mathbf{Z}}(n - \epsilon_n, n + \epsilon_n))$ is a FSN for

0_Y. Since sets of the form $\bigcup_{\mathbf{Z}}(n-\epsilon_n, n+\epsilon_n)$ are open, p_Y-stable by b) and for each open $U \supset \mathbf{Z}$ in $\mathbf{R}$ there is such a set contained in U (the FSN property), we get the desired claim in virtue of the definition of the quotient topology;

(e) again by definition of the quotient topology, to find such a FSN we can simply apply p (technically $p\times(=)$, but there is a natural homeomorphism) to a FS of stable open neighborhoods for the preimage of $p(0,0)$. But this is just $\mathbf{Z} \in \mathbf{R}^2$ with the subspace topology (not $\{0\} \times \mathbf{Z}$ with the product topology!), to be precise taken on the y-axis. Now, the preimage of the given system is just the union of arbitrarily small open squares around each integer, so indeed a collection of p_Y-stable, open supersets of $p^{-1}(p(0,0))$ satisfying the FSN property, and we are done;

(f) if it was, then the image of a FSN as in e) would be contained in some FSN as in d). Taking U as the former with $\delta_n = 1/n,\ n \neq 0$ and arbitrary δ_0 and ϵ_n's, gives $\phi(U) \subset \bigcup_{n\neq 0}(-1/n, 1/n) \times p_Y((n-\epsilon_n, n+\epsilon_n))$, which is clearly not contained in any FSN as in d) as those have a fixed positive range δ for the first coordinate;

(g) Y' would be a bouquet of countably many circles and Z' a bouquet fo countably many cylinders.

(5) (a) As in Exercise 2 we observe that we can prove openness of p on a basis for the euclidean topology. But since for any $x \in \mathbf{R}^n$ there is a neighborhood $U \ni x$ such that $p|_U$ is injective (we will prove it in the next point), p is a homeomorphism on the image on U, and so it is open.

Note preliminarly that, as a discrete set in $\mathbf{R}^n$, $H_{n,k}$ is closed. Let $e_{k+1}, ..., e_n$ be the canonical basis vectors not involved in the quotient and let $S \subset \mathbf{R}^n$ be their span, which is isomorphic to $\mathbf{R}^{n-k}$. Then the graph of the equivalence relation is homeomorphic to $\mathbf{R}^{n-k} \times H^2_{n,k}$ (with the topologies induced by $\mathbf{R}^n$) as for any pair (x, y) in the graph there is a single $s \in S$ such that $x \sim s$, so the association $(x, y) \mapsto (s, x-s, y-x)$ is bijective (from (s, t, u) we recover (x, y) as $(s+t, s+t+u)$), continuous and open (s is given by a projection). So extending this map to a map $\mathbf{R}^{2n} to \mathbf{R}^{3n}$ with the same formula and s again given by projection onto S of the first factor, we obtain that the graph is closed as it is homeomorphic to the above product, which is closed as all of its factors are.

Exercise 3, d) immediately gives that $X_{n,k}$ is Hausdorff.

(b) C is the closed hypercube of sidelength $\frac{1}{2}$ in $\mathbf{R}^n$, so it is compact. The difference of two distinct elements $y, z \in C$ is a vector each of whose components (in the canonical basis) is $|y_i - z_i| \leq |y_i - x_i| + |x_i - z_i| \leq \frac{1}{2}$, so it cannot lie in S, as all nonzero vectors in S have at least one component larger than 1. Therefore, p is injective on C;

(c) we know that $X_{n,k}$ is connected (as a quotient of a connected space) and Hausdorff, so we just need to prove that for every $w \in X_{n,k}$, there is some open neighborhood W of w which is homeomorphic to an open subset of some euclidean space. But, as observed in a), this follows from the fact that for any $x \in \mathbf{R}^n$ there is a neighborhood of x on which p is injective, and hence an homeomorphism on the image (so we start from w, take x as any of its preimages, and the argument precisely gives us the homeomorphism between some $W \ni w$ and an open of $\mathbf{R}^n$);

(d) let us adopt the notation of Exercise 2 as follows: $X = \mathbf{R}^k$, $Y = \mathbf{R}^{n-k}$ and $\sim_X$ be the same relation of this Exercise but just on $\mathbf{R}^k$ and $\sim_Y$ be the equality relation. Then in the notation of Exercise 2 we have $Z = X \times Y = \mathbf{R}^n$ and that $\sim$ is precisely our relation (here the key, as in a), is that two vectors outside the span of the first k basis vector cannot be equivalent), so $Z' \simeq X_{n.k}$. Since p_X is open (it is identical to proving that p is open) and p_Y is too as the identity, by Exercise 2 we get that $X_{n,k} \simeq X' \times \mathbf{R}^{n-k}$.
Therefore, we just need to construct a homeomoprhism $\mathbf{R}^k/H_k \longrightarrow (S_1)^k$. But observe that we can again apply Exercise 2 to, say, $X = \mathbf{R}$ and $Y = \mathbf{R}^{k-1}$ with the equivalence relations again given by the canonical basis lattice, as we obtain the same relation on the product space by linear independence, so we can reduce the proof to the case $k = 1$, i.e. $\mathbf{R}/\mathbf{Z} \simeq S_1$, which is clearly true.