Chapter I

Introduction

1 - What is topology?

There are many different answers! One can say:

(1) it is the study of mathematical concepts around continuity, convergence, (connectedness, ...) local vs. global properties,
(2) or it has to do with "shapes" and how they can be deformed (in opposition to geometry, where things are much more rigid);
(3) it is a field of mathematics, established roughly in the late 1800's / early 1900, with its own problems / methods;
(4) it is a language / way of thinking about mathematics in general, and as such as important / comparable to analytic, algebraic, probabilistic, combinatorial ways of thinking
(In any mathematical field, it makes sense to look for topological ideas / interpretations).

History / dates: (1) Euler (graph theory)
(2) Cantor (subsets of $\mathbb{R}$)
(3) Jordan, Riemann, ... (subsets of $\mathbb{C}$)
(4) Poincaré (3-dim. and more)
(5) Hausdorff, Brouwer
(6) Fréchet, Banach
...

2 - Some topological results

We use here the notion of connected set / continuous function on subsets of $\mathbb{R}^n$; the definition, etc, will reappear later.

Th. 1 - (Jordan Curve Theorem)

Let $f: [0,1] \longrightarrow \mathbb{C}$ be continuous and assume that $f(0) = f(1)$ and f is injective on $[0,1[$. Then $\mathbb{C} - f([0,1])$ is the union of two disjoint (connected non-empty open sets, the "interior" which is bounded and the "exterior", which is unbounded.

(More precisely: $\exists\ \varphi : \mathbb{C} \longrightarrow \mathbb{C}$, bijective, continuous, φ^{-1} continuous, s.t. $\varphi(f([0,1]))$ is the unit circle)

Th. 2 - (Brouwer fixed point theorem)

Every $f: D(0,1) \longrightarrow D(0,1)$ continuous has at least one fixed point.

(True also for $f: K \longrightarrow K$ where $K \neq \emptyset$ is a compact convex subset of $\mathbb{R}^n$ for some n)

(Inscribed rectangle problem; Vaughan)

Th. 3 - Let $f: [0,1] \longrightarrow \mathbb{C}$ be continuous, with $f(0) = f(1)$ ~~and~~ and f injective on $[0,1[$. There are four distinct points x, y, z, w s.t. $f(x), f(y), f(z), f(w)$ are the corners of a rectangle in the plane.

Th. 4. (Poincaré Conjecture; Perelman)

Let X be a 3-dimensional compact connected oriented manifold ~~of dimension~~. If X is also simply-connected, then X "is" a 3-dimensional sphere,

$$X \text{ "="} \left\{ (x,y,z,w) \in \mathbb{R}^4 \mid x^2+y^2+z^2+w^2 = 1 \right\}$$

These results are quite advanced. It is also useful to see how some important theorems of Analysis I will be interpreted as topological facts:

(Extremum theorem) $f:[a,b] \longrightarrow \mathbb{R}$ continuous has a maximum/minimum	$f: X \longrightarrow Y$ continuous, X compact $\Rightarrow f(X)$ compact
(Intermediate value theorem) $f: I \longrightarrow \mathbb{R}$ continuous, where I is an interval $\Rightarrow$ $f(I)$ is an interval	$f: X \longrightarrow Y$ continuous, X ~~compact~~ connected $\Rightarrow f(X)$ connected
(Inverse bijection theorem) $f: I \rightarrow \mathbb{R}$ continuous and strictly increasing ~~bijection~~ $\Rightarrow$ $f^{-1}: f(I) \rightarrow I$ is continuous	NO GENERAL FORM!
(Bolzano–Weierstrass) Any bounded sequence (a_n) has a convergent subsequence	A bounded subset of $\mathbb{R}$ is relatively compact / If X is compact and $x_n \in X$ then (x_n) has a convergent subsequence.

3 - Prerequisites / notation

We recall some notation (and fix these notation for the remainder of the lecture).

$\mathbb{R}^n$, $\mathbb{C}^n$, $[a,b]$, $]a,b]$ $\left(= (a,b] \text{ in many books}\right)$

$$]a,b[= \{ x \in \mathbb{R} \mid a < x < b \}$$