Chapter II

Topological spaces

1 _ Definition

Part of the idea of topology is to define/speak of points of a set being close, sequences converging, etc. In $\mathbb{R}^n$, this is formalized by looking at the distance between points, ~~one~~ and abstracting this leads to <u>metric spaces</u>. However, analysis reveals that it is <u>not wise</u> to just try to impose a unique notion of closeness: for functions $f : [a,b] \longrightarrow \mathbb{R}$, being "close" might mean many things:

(1) ~~[illegible]~~ $f(x)$, $g(x)$ are "close" for all x
($\rightsquigarrow$ "pointwise" convergence)

(2) $f(x)$, $g(x)$ are close <u>uniformly</u>
($\rightsquigarrow$ uniform convergence)

(3) $f(x)$, $g(x)$ are close "<u>on average</u>"
($\rightsquigarrow$ convergence in mean)

Among these, (1) cannot be defined using a <u>distance</u>, but (2) and (3) can, but give different convergent sequences. Moreover, one discovers after a bit of work and experience that (1) cannot be studied very well using just sequences and convergent sequences.

This led in the end to abandoning the idea of <u>describing</u> all ways to make sense of ideas of "closeness", and instead to a completely abstract definition of "the data needed to talk about topological ideas".

There are different equivalent ways of doing this. The most commonly used today is to specify which sets are <u>open</u> (intuitively: these are the sets which contain "all points sufficiently close" to any of their elements).

<u>Definition</u> - [Topological spaces; topology]

(1) Let X be a set (arbitrary). A <u>topology</u> on X is a set $\mathcal{T}$ of "open" subsets $U \subset X$ [i.e. a subset of the set $\mathcal{P}(X)$ of all subsets of X] such that the following conditions hold:

(1) $\emptyset$, X are open (i.e. $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$)

(2) If I is a <u>finite</u> set and U_i is open for $i \in I$, then $\bigcap_{i \in I} U_i$ is open [i.e. is in $\mathcal{T}$]

(3) If I is <u>any</u> set and U_i is open for all $i \in I$, then $\bigcup_{i \in I} U_i$ is open.

(2) A <u>topological space</u> is $(X, \mathcal{T})$ where $\mathcal{T} \subset \mathcal{P}(X)$ is a topology on X

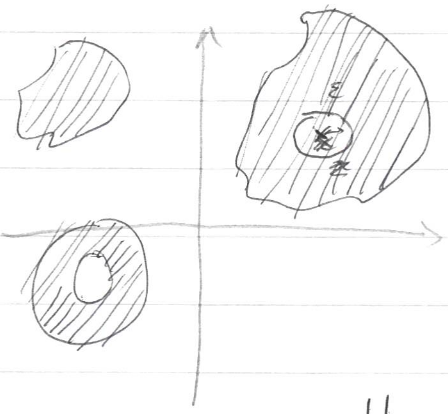
(One says that $\mathcal{T}$ is stable by finite intersections and arbitrary union)

<u>Notation</u>: if $Y \subset X$ and $X - Y$ is open, ~~we~~ one says that Y is <u>closed</u>.

<u>Example</u> - Let $X = \mathbb{R}^n$, where $n \geq 0$ is an integer.

The "ordinary"/"euclidian" topology is defined by the definition that $U \subset \mathbb{R}^n$ is <u>open</u> if and only if, for all $x \in U$, there exists $\varepsilon > 0$ such that the (euclidean) ball centered at x of radius ε is contained in U:

$\forall x \in U, \exists \varepsilon > 0, \forall (y_1, \dots, y_n) \in \mathbb{R}^n, \left(\sum |x_i - y_i|^2\right)^{1/2} < \varepsilon$
$\Rightarrow x + y \in U.$



The conditions (1), (2), (3) are elementary in that case: in particular, for (2), if $x \in \bigcap U_i$ and $\varepsilon_i > 0$ "works" for $x \in U_i$, then $\varepsilon = \min_{i \in I}(\varepsilon_i)$ "works" for $\bigcap U_i$.
Note here that if we allow I to be infinite, then this doesn't work any more (e.g.

$$\bigcap_{n \geq 1} \left] -\frac{1}{n}, \frac{1}{n} \right[= \{0\}$$

is not open).

We will see many more examples soon, but it is immediately useful to define the continuous maps in general, as these are the ways that different spaces can be related/compared.

Definition – Let X, Y be topological spaces. A map $f : X \longrightarrow Y$ is continuous [with respect to the given topologies] if and only if
$\forall U \subset Y$ open, $f^{-1}(U) \subset X$ is open.
($\Leftrightarrow \forall C \subset Y$ closed, $f^{-1}(C) \subset X$ is closed)
If f is furthermore bijective and f^{-1} is continuous then f is called a homeomorphism.

(This means $U \subset Y$ open $\Leftrightarrow f^{-1}(U)$ open]

Example – $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous (for the euclidian topologies) if and only if it is continuous in the sense of Analysis I+II:
(*) $\forall x \in \mathbb{R}^n, \forall \varepsilon > 0, \exists \delta > 0, \forall y \in \mathbb{R}^n,$
$$d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$$

(with $d(\cdot,\cdot)$ being the euclidian distance).

<u>Proof</u>: (1) Suppose f is continuous in the sense of the definition above.

Let $x \in \mathbb{R}^n$ be given, $\varepsilon > 0$.

The set $U = \{ z \in \mathbb{R}^m \mid d(f(x), z) < \varepsilon \} \subset \mathbb{R}^m$ is open (exercise); by definition,

$$V = f^{-1}(U) = \{ y \in \mathbb{R}^n \mid f(y) \in U \}$$

is open.

We have obviously $x \in V$ (since $d(f(x), f(x)) = 0$) so by definition there exists $\delta > 0$ s.t. the ball of radius δ around x is contained in V; this means that $d(x, y) < \delta \Rightarrow y \in V \Rightarrow d(f(x), f(y)) < \varepsilon$.

(2) Conversely, assume f satisfies $(*)$, and let $U \subset \mathbb{R}^m$ be open. Let $V = f^{-1}(U)$, and let $x \in V$ be given.

Since $\left\{ \begin{array}{l} U \text{ is open} \\ f(x) \in U \end{array} \right.$, there is an $\varepsilon > 0$ such that

$$d(f(x), z) < \varepsilon \Rightarrow z \in U.$$

By $(*)$, we find $\delta > 0$ s.t. $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$.

In particular, $d(x, y) < \delta \Rightarrow f(y) = z \in U$. So

$$\{ y \in \mathbb{R}^n \mid d(x, y) < \delta \} \subset V$$

which means that U is open.

$\square$

<u>Definition</u>. Let $(X, \mathcal{T})$ be a topological space. For $x \in X$, any set $Y \subset X$ such that $x \in Y$ and Y contains an <u>open set</u> U with $x \in U$ is called a <u>neighborhood</u> of x.

<u>Note</u>: A neighborhood is <u>not</u> always open.

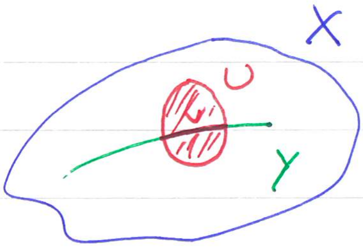
2 - Examples

As always when learning new definitions or concepts, it is very important to build up a store of examples which can be used to understand the topic. We will do so now for topological spaces (omitting many small checks, some of which will be in exercises).

2.1 - Subspaces

Let X be a topological space, and let $Y \subset X$ be any subset. The subspace topology on Y is defined by

$U \subset Y$ is open $\iff$ there is an open set $V \subset X$ such that $U = V \cap Y$.



[in other words: $\mathcal{T}_Y = \{ U \cap Y \mid U \in \mathcal{T}_X \}$]

To check that this is a topology, note that

$$\emptyset = Y \cap \emptyset \quad , \quad Y = X \cap Y$$

$$\bigcup_i (Y \cap U_i) = Y \cap \left(\bigcup_i U_i \right)$$

$$\bigcap_i (Y \cap U_i) = Y \cap \left(\bigcap_i U_i \right)$$

The inclusion map $j : Y \longrightarrow X$ is then continuous: for $V \subset X$ open, $j^{-1}(V) = Y \cap V$ is open in Y by definition.

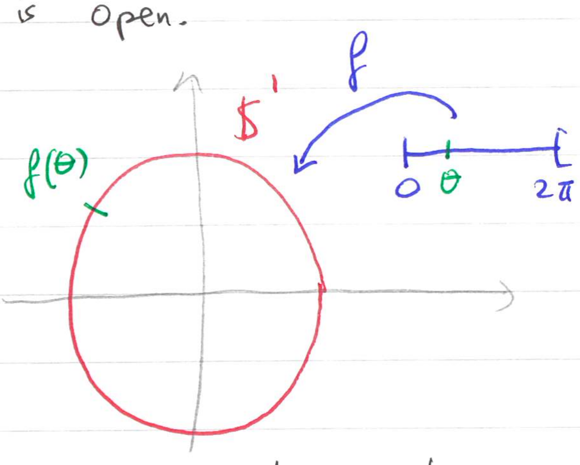
Moreover, if $f : X \longrightarrow Z$ is continuous then $f|Y$ is also continuous since $f|Y = f \circ j$, and:

Proposition. If $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$ are continuous, then $f_2 \circ f_1$ is continuous.

Proof. $$(f_2 \circ f_1)^{-1}(\underset{\displaystyle X_3}{\underset{\cap}{W}}) = \underbrace{f_1^{-1}\big(\underbrace{f_2^{-1}(W)}_{\text{open in } X_2 \text{ since } f_2 \text{ continuous}}\big)}_{\text{open in } X_1 \text{ since } f_1 \text{ continuous}} . \qquad \square$$

Ex. For intervals I in $\mathbb{R}$, we recover the usual definition of continuous functions $I \longrightarrow \mathbb{R}^m$ for instance.

As a specific example, let
$$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$
be the unit circle. It has then a topology where, for instance
$$\{(x,y) \mid x^2+y^2 = 1, \ |y| < \tfrac{1}{2}\} = S^1 \cap \{(x,y) \mid |y| < \tfrac{1}{2}\}$$
is open.



But moreover consider the parameterization
$$\begin{cases} [0, 2\pi[\xrightarrow{\ f\ } S^1 \\ \theta \longmapsto (\cos(\theta), \sin(\theta)) \end{cases}$$

This map is injective, surjective, and continuous. But $f^{-1} : S^1 \longrightarrow [0, 2\pi[$ is <u>not</u> continuous.

For instance, $I = [0, \frac{\pi}{2}[\ \subset [0, 2\pi[$ is <u>open</u> (since it is $]-\frac{\pi}{2}, \frac{\pi}{2}[\cap [0, 2\pi[$, for instance) and
$$(f^{-1})^{-1}(I) = f(I) =$$ 
is <u>not</u> open in S^1.

2.2 - Discrete topologies

If X is *any* set, we can define the *discrete topology* on X by saying that *any* subset is open. In particular, *any* map $f : X \longrightarrow Y$ is continuous (but not $f : Y \longrightarrow X$ in general!) in that case, whatever Y and its topology are.

Ex. $X = \mathbb{R}_d = (\mathbb{R}$, discrete topology$)$

The identity $\mathbb{R}_d \longrightarrow \mathbb{R}_{euclidean}$ is continuous (and bijective) but it is *not* a homeomorphism:

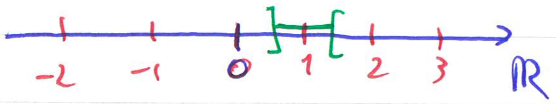
$$\mathbb{R}_{eucl} \longrightarrow \mathbb{R}_d$$

is *not* continuous (e.g. the inverse image of $\{0\}$, which is open in $\mathbb{R}_d$, is $\{0\}$, which is *not* open in $\mathbb{R}_{eucl}$.).

On the other hand, $\mathbb{Z}_{eucl} = (\mathbb{Z}$, subspace topology of $\mathbb{Z} \subset \mathbb{R})$ is homeomorphic to $\mathbb{Z}_{disc}$.

(Indeed, any $\{x\} \subset \mathbb{Z}_{eucl}$ is open, as

$$\{x\} = \left]x - \tfrac{1}{2}, x + \tfrac{1}{2}\right[\cap \mathbb{Z}.)$$



2.3 - Metric spaces

$\mathbb{R}^n_{eucl}$ is an example (as are its subsets) of a topological space where the topology is defined by a metric (distance): a function $X \times X \longrightarrow [0, +\infty[$ which indicates how close two points are.

Def. X set. A *distance* on X is a function

$$d : X \times X \longrightarrow [0, +\infty[$$

such that

(i) $d(x,x) = 0$ and $d(x,y) = 0 \iff x = y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,z) \leq d(x,y) + d(y,z)$

The topology associated to d is defined by: $U \subset X$ is open if and only if, for all $x \in U$, there exists $\varepsilon > 0$ such that

$$d(x,y) < \varepsilon \Rightarrow y \in U.$$

We say that (X,d) is a metric space.

If (X, d) is a metric space, then for any $Y \subset X$, the restriction of d to $Y \times Y$ is also a distance on Y, and the topology on Y associated to $d|_{Y \times Y}$ is also the subspace topology.

Ex. $\mathbb{R}^n$ with $d(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2}$ is a metric space.

There are however many other distances on $\mathbb{R}^n$ which define the same topology: for instance

$$d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}, \quad p \geq 1$$

$$d_\infty(x,y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

Two metrics defining the same topology are called equivalent.

Not all topologies can be defined by a metric!

2.4 - Topological manifolds

If we have a "nice" family of topological spaces, we can further define a class of top. spaces "locally homeomorphic" to one of these. The most important case is that of (topological) manifolds.

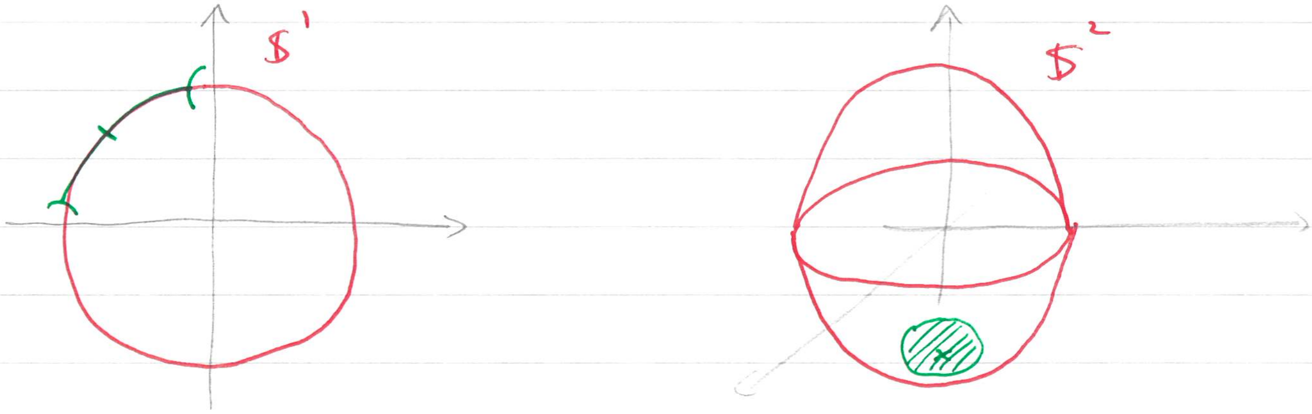
Definition. Let X be a topological space. One says that X is a topological manifold if every $x \in X$ has an open neighborhood $\mathcal{U}$ which is homeomorphic to an open set in some $\mathbb{R}^n$, $n \geq 0$ [where n may depend on x].

Ex. (1) $\mathbb{R}^n$

(2) $\mathbb{S}^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$

is a topological manifold.

Bad notation! It is not $(\mathbb{S}^1)^n$ ...



Note. (1) Some authors have a slightly different definition where X is also assumed to be separated / hausdorff (e.g. Hatcher, p° 47).

(2) If one can take the same value of n for all x in the definition, one says that X is an n-dimensional manifold. It is true (but very far from easy to prove) that n is uniquely determined.

(There exist continuous surjective maps $\mathbb{R} \longrightarrow \mathbb{R}^n$ for all n)

$\mathbb{R}^n$ and $\mathbb{S}^n$ are n-dimensional.

2.5 - Cantor set / space

Definition - Let $C = \{ (x_n)_{n \geq 1} \mid x_n \in \{0,1\} \text{ for all } n \}$

Define a topology on C by saying that $U \subset C$ is open if and only if for every $x \in U$, there exists a finite set I of positive integers [which may depend on x] such that

$$\{ (y_n)_{n \geq 1} \mid y_i = x_i \text{ for } i \in I \}$$

is contained in U.

(Later we will see this as an example of product topology)

Define

$$b: C \longrightarrow [0,1]$$
$$(x_i) \longmapsto \sum_{i=1}^{+\infty} \frac{x_i}{2^i}$$

$< \frac{1}{2} + \frac{1}{4} + \ldots = 1$

This map is surjective and one can check that it is continuous when $[0,1]$ has the subspace topology from $[0,1] \subset \mathbb{R}$. It is not a homeomorphism.

On the other hand define

$$t: C \longrightarrow [0,1]$$
$$(x_i) \longmapsto \sum_{i=1}^{+\infty} \frac{2x_i}{3^i}$$

Then t is injective, continuous and $t: C \longrightarrow t(C) \subset [0,$ is a homeomorphism. The image $t(C) \subset [0,1]$ is called the "middle third" Cantor set: it is the set of real numbers in $[0,1]$ whose ternary (= base 3) expansion only has digits 0 and 2.

2.6 - Functions

<u>Definition</u> - Let X be a set and $\mathcal{F}(X)$ the set of <u>all</u> functions $f : X \longrightarrow \mathbb{C}$. If X has a topology, let $\mathcal{C}(X) \subset \mathcal{F}(X)$ be the subset of <u>continuous</u> functions (with the euclidean topology on $\mathbb{C}$).

(1) The <u>topology of pointwise convergence</u> $\mathcal{T}_p$ on $\mathcal{F}(X)$ has $U \subset \mathcal{F}(X)$ open $\iff$ for all $f \in U$, there exists a <u>finite</u> set $S \subset X$ and $\delta_x > 0$ for $x \in S$ such that

$$\{ g : X \longrightarrow \mathbb{C} \ / \ \text{for } x \in S, \ |f(x) - g(x)| < \delta_x \} \subset U$$

(2) The <u>topology of uniform convergence</u> $\mathcal{T}_u$ on $\mathcal{F}(X)$ has $U \subset \mathcal{F}(X)$ open $\iff$ for all $f \in U$, there exists $\delta > 0$ such that

$$\{ g : X \longrightarrow \mathbb{C} \ / \ \text{for } \underline{\text{all}} \ x \in X, \ |f(x) - g(x)| < \delta \} \subset U.$$

If $U \subset \mathcal{F}(X)$ is open for $\mathcal{T}_p$, it is also open for $\mathcal{T}_u$, but not conversely in general. [Suppose $f \in U$ and $S \subset X$ is finite, $\delta_x > 0$ for $x \in S$, s.t. $(\forall x \in S, \ |f(x) - g(x)| < \delta_x \Rightarrow g \in U)$; take $\delta = \min_{x \in S} \delta_x > 0$, then $(\forall x \in X, \ |f(x) - g(x)| < \delta \Rightarrow g \in U)$.]

One can show that $\mathcal{C}(X) \subset \mathcal{F}(X)$ is <u>closed</u> for $\mathcal{T}_u$, but not for $\mathcal{T}_p$ in general (abstract version of: a uniform limit of continuous functions is continuous, but not in general a pointwise limit).

The evaluation maps $ev_x \left\{ \begin{array}{l} \mathcal{F}(X) \longrightarrow \mathbb{C} \\ f \longmapsto f(x) \end{array} \right.$ for $x \in X$ are all continuous for $\mathcal{T}_p$,
and then for $\mathcal{T}_u$ [since the identity $(X, \mathcal{T}_u) \longrightarrow (X, \mathcal{T}_p)$ is itself continuous.] Indeed let $V \subset \mathbb{C}$ be open; then $ev_x^{-1}(V) = \{ f : X \longrightarrow \mathbb{C} \ / \ f(x) \in V \}$; let $f \in ev_x^{-1}(V)$; since $f(x) \in V$, there is $\delta > 0$ s.t. $(|z - f(x)| < \delta \Rightarrow z \in V)$, and then

$$ev_x^{-1}(V) \supset \{ g : X \longrightarrow \mathbb{C} \ / \ |g(x) - f(x)| < \delta \} \quad .]$$

2.7 – Topology and other structures

Topology is so important that whenever one has a set with some other type of "structure", it makes sense to also look at adding a "compatible" topology.

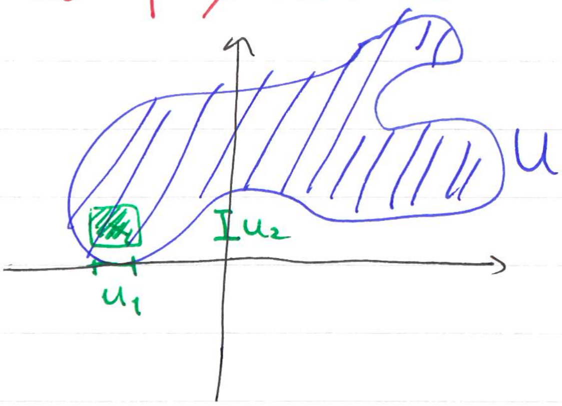
For instance, we have the notion of topological group:

Definition. A topological group is the data of a set G with operations

$$\cdot : G \times G \longrightarrow G \quad \text{and} \quad e \in G$$
$$^{-1} : G \longrightarrow G$$

and a topology $\mathcal{T}$ such that the group axioms are satisfied and the multiplication and inverse are continuous

[$G \times G$ being given the product topology, which will be discussed later: $U \subset G \times G$ is open if for every $(x,y) \in U$, there exist open neighborhoods U_1 of x and U_2 of y such that $U_1 \times U_2 \subset U$.]



For example, $\mathbb{R}$ with the usual addition and euclidian topology is a topological group.

However, $\mathbb{R}$ with the topology where a set is open if it is either empty or has finite complement is not a topological group.

3 – Basis

In a topology, there is often a lot of "redundancy" which can be avoided for certain purposes, such as checking that a function is continuous.

Definition – Let $(X, \mathcal{T})$ be a topological space. A subset $\mathcal{B} \subset \mathcal{T}$ is a basis of $\mathcal{T}$ if any open set $U \subset X$ is a union of sets from $\mathcal{B}$.

Lemma – $f : X \longrightarrow Y$ is continuous $\iff$ for all open sets $U \subset Y$ in a basis of the topology of Y, the inverse image $f^{-1}(U) \subset X$ is open.

Proof – $\Rightarrow$ is because any basis element is open by definition.
$\Leftarrow$ Let $U \subset Y$ be open; by assumption we can find $(U_i)_{i \in I}$, with U_i in the basis, such that

$$U = \bigcup_{i \in I} U_i$$

in which case

$$f^{-1}(U) = \bigcup_{i \in I} f^{-1}(U_i)$$

$\left(= \{ x \in X \mid \exists i, f(x) \in U_i \}\right)$

and this is a union of open sets in X, so is open.

$\square$

In a similar way, observe that $U \subset X$ is open $\iff$ it contains a neighborhood of every $x \in U$. [$\Rightarrow$ U is a neighborhood of every $x \in U$; $\Leftarrow$: let $V_x \subset U$ be an open neighborhood of any given $x \in U$; then

$$U = \bigcup_{x \in U} V_x$$

is a union of open sets, so is open in X.]

<u>Definition</u> - A <u>fundamental system</u> of neighborhoods of $x \in X$ is a set $\mathcal{V}_x$ of neighborhoods of x such that for any neighborhood W of x, there is a $V \in \mathcal{V}_x$ such that $V \subset W$.

<u>Lemma</u> - Let $f: X \to Y$ be a map.
The following are equivalent:
(1) f is continuous
(2) for all $x \in X$, for all neighborhood V of $f(x)$ in Y, there is a neighborhood U of x in X such that $f(U) \subset V$
(3) for all $x \in X$, for all element V of a fundamental system of neighborhoods of $f(x)$ in Y, there is an element U of a fundamental system of neighborhoods of x in X such that $f(U) \subset V$.

<u>Proof</u> - (2) $\Rightarrow$ (3) : elementary
(3) $\Rightarrow$ (2) : let V be a neighborhood of $f(x)$ in Y; there is a $V_1 \subset V$ in the fundamental system, and there is a $U_1 \subset X$ with $f(U_1) \subset V_1 \subset V$.
(1) $\Rightarrow$ (2) : let $x \in X$, V a neighborhood of $f(x)$, $V_1 \subset V$ an open neighborhood of $f(x)$; by continuity, $U = f^{-1}(V_1)$ is open, and contains x, so is a neighborhood of x with $f(U) \subset V_1 \subset V$.
(2) $\Rightarrow$ (1) : let $V \subset Y$ be open; let $x \in f^{-1}(V)$; then V is an open neighborhood of $f(x)$, so there is a $U \subset X$ neighborhood of x with $f(U) \subset V$, hence $U \subset f^{-1}(V)$.

So $f^{-1}(V)$ contains an open neighborhood of each of its points, so it is open in X (cf. remark p° 16).

$\square$

[redacted]

<u>Examples</u> - (1) In $\mathbb{R}$, the intervals $]a,b[$ form a basis of the euclidean topology.

The intervals $]a,b[$ with a, b in $\mathbb{Q}$ also form a basis of the topology.

The intervals $\left]x_0 - \frac{1}{2^k}, x_0 + \frac{1}{2^k}\right[$, $k \geq 1$, form a fundamental system of neighborhoods of x_0.

(2) In $\mathbb{R}^n$, taking the open balls

$$\{x \in \mathbb{R}^n \mid d_{eucl}(x, x_0) < \varepsilon\}$$

for $\varepsilon > 0$ as fundamental system of neighb. of x_0, we recover the result of page 7.

(3) In any metric space (X, d), the open balls

$$B^d(x_0; \varepsilon) = \{x \in X \mid d(x, x_0) < \varepsilon\}, \quad \varepsilon > 0$$

form a fundamental system. This is also true of the $B^d(x_0; \frac{1}{n})$, $n \geq 1$, and in particular, there is a <u>sequence</u> (or countable set) of open neighb. of x_0 that forms a fund. system.

(4) In any topological manifold, there is a countable fund. system of open neigh. of any point (since there is a neighb homeomorphic to an open set of some $\mathbb{R}^n$).