4 – Closure, interior, boundary

Let $(X, \mathcal{T})$ be a topological space.

Definition – For $A \subset X$ an arbitrary subset, we define:

(1) The <u>interior</u> $\mathring{A}$ of A, as the <u>union</u> of all open sets contained in A:

$$\mathring{A} = \bigcup_{\substack{U \subset A \\ U \text{ open in } X}} U$$

This is an open set (union of open sets), the largest (for inclusion) open set contained in A.

(2) The <u>closure</u> $\bar{A}$ of A, as the intersection of all open set containing A:

$$\bar{A} = \bigcap_{\substack{C \supset A \\ C \text{ closed}}} C \qquad (*)$$

This is a closed set, the largest containing A.

(3) The <u>boundary</u> $\partial A = \bar{A} \cap \overline{(X - A)}$. It is closed.

(4) A is <u>dense</u> in X if $\bar{A} = X$.

The following properties are then easy consequences of the definitions:

(1) $\widehat{X - A}^{\,\circ} = X - \bar{A}$,

(2) $\overline{X - A} = X - \mathring{A}$,

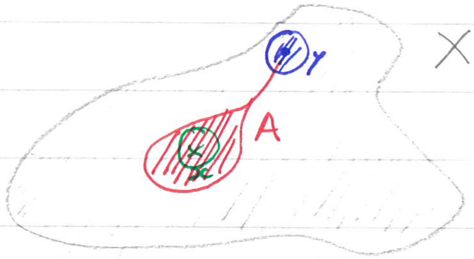
(3) $\partial A = \bar{A} - \mathring{A}$.

Lemma – Let $A \subset X$ be a subset, and $x \in X$.

(1) We have $x \in \mathring{A}$ if and only if there exists an open set U s.t. $x \in U$ and $U \subset A$.

(2) We have $x \in \bar{A}$ if and only if every (open) neighb. of x intersects A, if and only if there is no open set U s.t. $x \in U$ and $U \cap A = \emptyset$.

(3) We have $x \in \partial A$ if and only if all (open) neighb. of x intersect both A and $X - A$.



$x \in \mathring{A}$

$y \in \bar{A}$

<u>Proof.</u> (1) If $x \in \mathring{A}$ then there exists $U \subset A$ open with $x \in U$ by definition.

(2) Similar for the closure.

(3) Same: it is a transcription of the definition.

$\square$

<u>Remarks and Examples.</u> (1) $\bar{A} = A$ if and only if A is closed and similarly $\mathring{A} = A$ if and only if A is open. (Indeed, $A \subset \bar{A}$ by definition; if $\bar{A} = A$ then A is closed since $\bar{A}$ is; if A is closed then $\bar{A} \subset A$ since we can take $C = A$ as a closed set in the definition (*)).

(2) Let $I =]a, b] \subset \mathbb{R}$. Then $\bar{I} = [a, b]$, and $\mathring{I} =]a, b[$, and $\partial I = \{a, b\}$.
(<u>Note</u>: if I has the subspace topology then $\bar{I} = \mathring{I} = I$ since I is open and closed in the subspace topology.)

(3) If X is discrete then $\mathring{A} = \bar{A} = A$ for all $A \subset X$ since every set is open and closed.

(4) $X = \mathring{A} \cup \overbrace{(X - A)}^{\circ} \cup \partial A$, where the union is disjoint, and $A = \mathring{A} \cup \partial A$, where the union is disjoint.

(5) Let $A = \mathbb{Q} \subset \mathbb{R}$. Then

$$\begin{cases} \mathring{A} = \emptyset \\ \overline{A} = \mathbb{R} \\ \partial A = \mathbb{R} \end{cases} \qquad (\Leftrightarrow \mathbb{Q} \text{ is dense in } \mathbb{R}).$$

(6) In a metric space (X, d), we have

$$\overline{B(x_0, \delta)} \subset \{ x \in X \mid d(x, x_0) \leq \delta \} \quad (**)$$

but there is not always equality.

[Ex. define $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$, then

$$B(x_0, 1) = \{x_0\} \quad \text{so} \quad \overline{B(x_0, 1)} = \{x_0\}$$

but $\{ x \in X \mid d(x, x_0) \leq 1 \} = X$.]

More generally: for any continuous map $f: X \longrightarrow Y$ and $A \subset Y$, we have

$$\overline{f^{-1}(A)} \subset f^{-1}(\overline{A}).$$

(Taking the continuous function

(Check!)
$$f: X \longrightarrow [0, +\infty[$$
$$x \longmapsto d(x, x_0)$$

and $A = [0, \delta[$, we obtain $(**)$.)

Indeed: for any closed set C containing A in Y, we get

$$A \subset C \Rightarrow f^{-1}(A) \subset f^{-1}(C)$$

so

$$f^{-1}(A) \subset \bigcap_{C \supset A} f^{-1}(C) = f^{-1}\Big(\bigcap_{C \supset A} C\Big)$$
$$= f^{-1}(\overline{A});$$

the right-hand side is a closed set containing $f^{-1}(A)$, so it contains its closure $\overline{f^{-1}(A)}$.

5 – Limits and convergence

Definition – Let $(X, \mathcal{T})$ be a topological space. Let $(x_n)_{n \geq 1}$ be a sequence of elements of X. Let $y_0 \in X$. One says that x_n converges to y_0 [for $\mathcal{T}$], denoted $\lim_{n \to \infty} x_n = y_0$, or $x_n \longrightarrow y_0$, if:

for all (open) neighborhood U of y_0, there exists $N \geq 1$ such that $x_n \in U$ for all $n \geq N$.

(One can restrict U to a fundamental system of neighb. of y_0)

Ex. (1) In $\mathbb{R}^n$, this is the same as "the usual definition". Similarly, in a metric space (X, d) we have $x_n \longrightarrow y_0$ if and only if

$$\lim_{n \to \infty} d(x_n, y_0) = 0.$$

(limit in $\mathbb{R}$)

(if and only if: $\forall \varepsilon > 0$, $\exists N$, $\forall n \geq N$, $d(x_n, y_0) < \varepsilon$.)

(2) Let $C = \{(x_n)_{n \geq 1} \mid x_n \in \{0, 1\}\}$ be the Cantor space.

Let $x^{(m)} = (x_n^{(m)})$ be in C for $m \geq 1$. Then $x^{(m)} \longrightarrow y = (y_n)$ if and only if (for all $n \geq 1$,) $x_n^{(m)} = y_n$ for all m large enough [which may depend on n].

(Indeed, $U_N = \{x \in C \mid x_n = y_n \text{ for } n \leq N\}$ is an open neighb. of y, and any neigh. of y contains U_N for some N.)

(3) Let X be a topological space and (f_n) a sequence of functions $f_n : X \longrightarrow \mathbb{C}$. Let $g : X \longrightarrow \mathbb{C}$.

- The sequence (f_n) converges to g for $\left\{\begin{array}{l}\text{the } \cancel{\text{pology}} \text{ topology of pointwise convergence} \\ \text{the topology of uniform convergence}\end{array}\right\}$ if and only if $\left\{\begin{array}{l}\text{for all } x_0 \in X,\ (f_n(x_0)) \text{ converges in } \mathbb{C} \text{ to } g(x_0) \\ f_n \text{ converges to } g \text{ uniformly on } X\end{array}\right\}$.

<u>Proof.</u> [For $\mathcal{T}_p$] (1) Suppose $f_n \longrightarrow g$ for $\mathcal{T}_p$. Pick $x_0 \in X$. Let $\varepsilon > 0$. The set

$$\{ f : X \to \mathbb{C} \mid |f(x_0) - g(x_0)| < \varepsilon \} \subset \mathcal{F}(X)$$

is <u>open</u> for $\mathcal{T}_p$ [check!] and contains g, so it contains f_n for n large enough, so $|g(x_0) - f_n(x_0)| < \varepsilon$ for n large enough, which means that $f_n(x_0) \longrightarrow g(x_0)$ in $\mathbb{C}$.

(2) Conversely, suppose $f_n(x_0) \longrightarrow g(x_0)$ for all $x_0 \in X$. Let $U \subset \mathcal{F}(X)$ be open containing g. By definition [p. 15] there exists $S \subset X$ <u>finite</u> and $\delta_x > 0$ for all $x \in S$ s.t.

$$V = \{ f : X \to \mathbb{C} \mid \forall x \in S,\ |f(x) - g(x)| < \delta_x \} \subset U$$

Since $f_n(x) \longrightarrow g(x)$ for all $x \in S$, there exist ~~integ~~ integers N_x s.t.

$$\forall x \in S,\ \forall n \geq N_x,\ |f_n(x) - g(x)| < \delta_x$$

and hence for $N = \max_{x \in S} N_x$ finite set! we get

$$\forall x \in S,\ \forall n \geq N,\ |f_n(x) - g(x)| < \delta_x$$

and so $f_n \in V$ for $n \geq N$.

□

In $\mathbb{R}^n$ (or metric spaces), sequences "characterize" the topology: a subset A is <u>closed</u> $\Longleftrightarrow$ any sequence (x_n) with $x_n \in A$ for all n which converges <u>in X</u> has limits <u>in A</u>.

This does <u>not</u> always hold for general topological spaces.

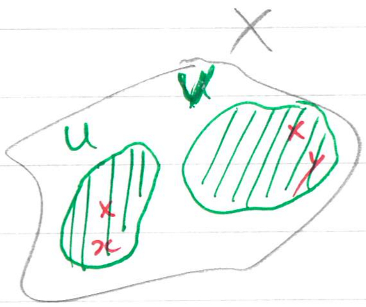
(1) It could be that "the" limit is <u>not</u> unique

(2) It could be that limits of sequence <u>do not</u> determine the topology: although any limit y of a sequence x_n with $x_n \in A$ belongs to $\bar{A}$ [because for any neigh. U of y, we have $x_n \in A \cap U$ for n large enough], the converse may <u>fail</u>: some $y \in \bar{A}$ may exist which is <u>not</u> the limit of <u>any</u> sequence in A.

However, we can get around this in various ways.

<u>Def.</u> [Hausdorff spaces] A topological space $(X, \mathcal{T})$ is <u>separated</u> or <u>Hausdorff</u> if for any $x \neq y$ in X there are neighb. U of x and V of y such that

$$U \cap V = \emptyset.$$



<u>Ex.</u> (1) <u>Any</u> metric space is separated: if $x \neq y$, let $\delta = d(x,y) > 0$; then the open balls

$$B^d\left(x, \frac{\delta}{2}\right) \text{ and } B^d\left(y, \frac{\delta}{2}\right)$$

are disjoint by the triangle inequality, and are open neighborhoods of x, y respectively.

(2) If X is infinite and $\mathcal{T} = \{\emptyset, \text{complements of finite sets}\}$ then X is <u>not</u> separated: if $x \neq y$ then any neighborhood U of x contains ~~infin~~ all except finitely elements of X (so maybe $y \notin U$), and similarly for V, neighb. of y, so $U \cap V$ contains ~~all~~ all but finitely elements and hence is not empty.

(3) Any subspace of a Hausdorff space is Hausdorff.

Proposition– If X is Hausdorff then limits of sequences are unique: if (x_n) converges to x and to y, then $x = y$.

Proof – If $x \neq y$, we can find U, V open with $U \cap V = \emptyset$ and $x \in U$, $y \in V$. For n large enough, however, we have $x_n \in U \cap V$ since $x_n \to x$ and $x_n \to y$, so we get a contradiction.

$\square$

In addition, we can avoid the Problem 2 under a finiteness condition:

Proposition – Let $(X, \mathcal{T})$ be a topological space. Assume that for all $x \in X$, there is a countable family $(U_{x,n})$ of neighb. of x which is a fundamental system. Then for any $A \subset X$, we have $x \in \bar{A}$ if and only if x is a limit of a sequence (x_n), with $x_n \in A$ for all n.

Proof – We need to show that if $x \in \bar{A}$, it is the limit of a sequence. ~~For each n, U_{x,n} ∩ A is not empty~~

To do this we observe that for each $n \geq 1$, the set

$$V_n = U_{x,1} \cap \cdots \cap U_{x,n}$$

is a neighborhood of x, and that $V_{n+1} \subset V_n$.

Moreover it is a fundamental system: if U is any neigh. of x, we find j s.t. $U_{x,j} \subset U$, and then $V_j \subset U_{x,j} \subset U$.

Now since $x \in \bar{A}$, we have $V_n \cap A \neq \emptyset$; let x_n be in $V_n \cap A$. We claim that $x_n \to x$: indeed, given U, neighb. of x, we find N so that $V_N \subset U$, and then for $n \geq N$, we have $x_n \in V_n \subset V_N \subset U$. So the definition of convergence holds.

$\square$

6 - Filters

To detect the topology using analogues of sequences, one can use two generalizations: "nets" or "filters". Filters are more general (and more relevant to other fields), although a bit less intuitive. We just give the basic definitions and facts.

Definition - Let X be a set. A *filter* on X is a set $\mathcal{F}$ of subsets of X such that:

(i) $\emptyset \notin \mathcal{F}$

(ii) if $A \in \mathcal{F}$ and $B \supset A$, then $B \in \mathcal{F}$

(iii) if ~~$A \in \mathcal{F}$ and~~ $(A_i)_{i \in I}$ is a *finite* family of elements of $\mathcal{F}$, then $\bigcap A_i$ is in $\mathcal{F}$ (it suffices to check this for $A_1 \cap A_2$)

Examples - (i) Let $(X, \mathcal{T})$ be a topological space. Let $x_0 \in X$. Then

$$\mathcal{F}_{x_0} = \{ A \subset X \mid A \text{ is a neighb. of } x_0 \}$$

is a filter on X.

(ii) Let X be an *infinite* set. Then

$$\mathcal{F} = \{ A \subset X \mid X - A \text{ is finite} \}$$

is a filter on X.

(iii) The set

$$\mathcal{F}_\infty = \{ A \subset \mathbb{R} \mid \exists t,\ [t, +\infty[\subset A \}$$

is a filter on $\mathbb{R}$.

(iv) Let (x_n) be a sequence in X. The <u>elementary filter</u> associated to (x_n) is

$$\mathcal{F} = \{ A \subset X \mid \exists N \geq 1, \forall n \geq N, \quad x_n \in A \}$$

(v) Let $f : X \longrightarrow Y$ be any map, and $\mathcal{F}$ a filter on X. The set

$$f_*(\mathcal{F}) = \{ A \subset Y \mid \exists B \in \mathcal{F}, \ A \supset f(B) \}$$

is a filter on Y.

<u>Definition</u>. Let $(X, \mathcal{T})$ be a topological space, $\mathcal{F}$ a filter on X and $x_0 \in X$.

(i) $\left\{ \begin{array}{l} \text{The filter } \mathcal{F} \text{ converges to } x_0 \\ \text{The point } x_0 \text{ is a } \underline{\text{limit}} \text{ of } \mathcal{F} \end{array} \right\}$ if $\mathcal{F}$ contains the filter of neighborhoods of x_0 (equivalently: for every neighborhood U of x_0, there exists $A \in \mathcal{F}$ s.t. $A \subset U$.)

(ii) Let $f : X \to Y$ be a map, where Y is a topological space. One says that an element $y \in Y$ is a <u>limit point of f along $\mathcal{F}$</u> if $f_*(\mathcal{F})$ converges to y.
($\Leftrightarrow \forall U$ neighb. of y, $\exists A \in \mathcal{F}$, $f(A) \subset U$.)

<u>Lemma</u>. (1) A sequence (x_n) converges to y [in the sense of page 22] if and only if the associated elementary filter $\underbrace{}_{\mathcal{F}}$ converges to y. [in the sense above]

(2) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function. We have

$$\lim_{x \to x_0} f(x) = y$$

(in the ε-δ sense) if and only if y is limit of f along the filter of neighborhoods of x_0.

This shows that these definitions generalize the classical ones.

Proof - (1) If $x_n \to x$ in the classical sense, then for any neighb. U of x, we have $x_n \in U$ for $n \geq N$ for some N, so

$$\underbrace{\{ x_n \mid n \geq N \}}_{\in \mathcal{F}} \subset U$$

so the filter $\mathcal{F}$ converges to x.

Conversely, if $\mathcal{F}$ converges to x, then for any U, we find $A \in \mathcal{F}$ contained in U, i.e, some set A s.t. for some N, $A \supset \{ x_n \mid n \geq N \}$.

So $x_n \in A$ ~~[illegible]~~ $\subset U$ for all $n \geq N$.

(2) Suppose $f(x) \to y$ in the $\varepsilon - \delta$ sense. Let U be a neighb. of y. There is a neighb. V of x_0 such that $f(V) \subset U$, (U contains a ball $B(y, \varepsilon)$, $\varepsilon > 0$, and $f^{-1}(U)$ contains a ball $V = B(x_0, \delta)$ s.t. $f(V) \subset B(y, \varepsilon) \subset U$, which means that $f_0(\mathcal{F}) \to y$, where $\mathcal{F}$ is the filter of neighb. of x_0.

Converse: exercise!

$\square$

Example - Filters have one big advantage on sequences: they unify all types of limits in analysis. For instance:

$$\lim_{x \to +\infty} f(x) = a \in \mathbb{R}$$

$$\Updownarrow$$

the filter $f_0(\mathcal{F}_\infty)$ converges to a, where

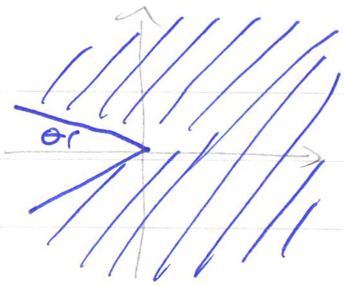
$\mathcal{F}_\infty = \{ A \mid \exists t,\ A \supset [t, +\infty[\}$

$$\lim_{x \to a} f(x) = +\infty$$

$$\Updownarrow$$

$\mathcal{F}_\infty \subset f_0(\mathcal{F}_a)$, where $\mathcal{F}_a$ is the filter of neigh. of a.

Here is a more fancy example: let

$$\mathcal{F} = \{ A \subset \mathbb{C} \mid \exists \theta > 0, \theta < \tfrac{\pi}{2},$$ 

$$\subset A \}$$

which is a filter on $\mathbb{C}$.

Then $\lim_{\mathcal{F}} f(x) = y$ means that $f(x)$ converges to y "as long as x doesn't get close to the negative real axis".

<u>Proposition</u> - Let X be a topological space and $A \subset X$. For $x \in X$, we have

$x \in \bar{A} \iff$ there is a filter $\mathcal{F}$ <u>on A</u> s.t. $i_*(A)$ converges to x, where $i : A \longrightarrow X$ is the inclusion.

This shows that filters and their limits <u>do</u> allow us to recover "all" the topology.

<u>Proof</u> - $\Rightarrow$: because $x \in \bar{A}$, the set

$$\mathcal{F} = \{ A \cap U \mid U \text{ neigh. of } x \}$$

is a filter on A (note $A \cap U \neq \emptyset$ for all U).

Now $i_*(\mathcal{F}) = \{ B \subset X \mid A \cap U \subset B$ for some U neighb. of $x \}$;

if U is a neighb. of x, then $U \cap A \subset U$ is an element of $\mathcal{F}$, so we get that $i_*(\mathcal{F})$ converges to x.

$\Leftarrow$: suppose $\mathcal{F}$ is a filter on A and $i_*(\mathcal{F})$ converges to x. Let U be a neighb. of x in X. By def. of convergence there is a set $B \in i_*(\mathcal{F})$ such that $B \subset U$. This means there is a $B' \subset A$ such that $B' \subset B \subset U$. Since $B' \neq \emptyset$, and $B' \subset U \cap A$, the intersection $U \cap A$ is not empty.

$\square$