Chapter III

Compactness, connectedness, completeness

These three properties are among the most important in topology; in Analysis I, they correspond to important ways to "construct" solutions of equations or solving problems:

(1) compactness of bounded closed intervals gives convergent subsequences

(2) connectedness of intervals gives the intermediate value theorem

(3) completeness of $\mathbb{R}$ ensures convergence of, e.g. bounded non-decreasing sequences.

In any topological space, such properties are also very useful

1 _ Compactness

In $\mathbb{R}$, compact sets are those which are, equivalently, closed and bounded or (Bolzano - Weierstrass) ~~conta~~ are such that every ~~conv~~ sequence contains a convergent subsequence. The first definition is too weak for applications ~~[illegible]~~ in general (and "bounded" is unclear ...), the second also insufficient since sequences do not "capture" all topological information. The Heine - Borel Property turns out to ~~be~~ the right property.

Definition - (1) A topological space $(X, \mathcal{T})$ is compact if for every open covering of X, there is a finite subcovering
In other words: if $(U_i)_{i\in I}$ is a family of open subsets

of X such that $X = \bigcup_{i \in I} U_i$, then we can find a finite subset $J \subset I$ such that

$$X = \bigcup_{i \in J} U_i$$

(2) A subset $A \subset X$ is compact if it is with the subspace topology. Equivalently: if $(U_i)_{i \in I}$ are open sets in X with $A \subset \bigcup_{i \in I} U_i$, there is a finite subset $J \subset I$ such that

$$A \subset \bigcup_{i \in J} U_i .$$

Note - In some books (eg French literature), "compact" also requires X to be Hausdorff, and the notion above is called "quasi-compact".

Proposition - Let $f : X \to Y$ be continuous, with X compact. Then $f(X) \subset Y$ is compact.

(But f^{-1}(compact set) is not always compact, ex. $f : \mathbb{R} \to \mathbb{R}$ constant)

Proof - If $f(X) \subset \bigcup_{i \in I} U_i$, $U_i \subset Y$ open, then $X = \bigcup_{i \in I} \underbrace{f^{-1}(U_i)}_{\text{open}}$, so we can find $J \subset I$ finite such that $X = \bigcup_{j \in J} f^{-1}(U_j)$, and then

$$f(X) \subset \bigcup_{j \in J} U_j ,$$

hence the result.

□

Lemma - (1) If $A \subset X$ is a compact subset of a hausdorff topological space, then A is closed.

(2) If X is compact and $A \subset X$ is closed, then A is compact.

(This is not always true for A closed in an arbitrary space!)

Note - The first part is false if X is not Hausdorff in general.
For instance, let $\mathcal{T}_{fin}$ be the cofinite topology on an infinite set X. It is compact: if $X = \bigcup U_i$, some U_{i_0} is not empty, so contains all but finitely many $x \in X$; taking one U_{i_x} for each $x \notin U_{i_0}$ we get

$$X = \bigcup_{i \in \{i_0\} \cup \{i_x\}} U_i.$$

But for the same reason, any subset of X is compact, although not all are open (e.g. $\mathbb{Z} \subset (\mathbb{R}, \mathcal{T}_{fin})$ is not open).

Proof - (1) We show that $X - A$ is open by showing that each $x_0 \in X - A$ is interior to $X - A$. To do this, we use the Hausdorff condition as follows: for each $x \in A$, there are open neighborhoods U_x of x_0 and V_x of x such that $U_x \cap V_x = \emptyset$.
Then

$$A \subset \bigcup_{x \in A} V_x$$

so there is a finite set $B \subset A$ s.t.

$$A \subset \bigcup_{x \in B} V_x.$$

Let $U = \bigcap_{x \in B} U_x$; since B is finite, this is an open neighborhood of x_0.
But now $U \cap A = \emptyset$, so $U \subset X - A$; indeed if $y \in U \cap A$, then $y \in U_x$ for all $x \in B$, but then $y \notin V_x$ for all $x \in B$ [since $U_x \cap V_x = \emptyset$], hence $y \notin \bigcup_x V_x$, a fortiori $y \notin A$.

(2) This is easier: let $(U_i)_{i \in I}$ be open such that

$$A \subset \bigcup U_i.$$

Then $X \subset \bigcup_{i \in I} U_i \cup \underbrace{(X - A)}_{open}$, so there is a finite set $J \subset I$ s.t. $X \subset \bigcup_{i \in J} U_i \cup (X - A)$; then $A \subset \bigcup_{i \in J} U_i$.
$\square$

The following is an <u>extremely useful</u> corollary.

<u>Proposition.</u> Let $f: X \longrightarrow Y$ be a continuous bijection. If X is compact and Y Hausdorff, then f is a <u>homeomorphism</u>

<u>Proof.</u> We need to check that $f^{-1}: Y \longrightarrow X$ is continuous, which means checking that $(f^{-1})^{-1}(A) = f(A)$ is closed in Y if $A \subset X$ is closed. But then A is compact (closed in a compact space) so $f(A) \subset Y$ is compact (image of a compact set), so it is closed (because Y is Hausdorff)

$\square$

<u>Examples.</u> (1) Any finite set with the discrete topology is compact (and Hausdorff). In fact:

<u>Prop.</u> Let X be a discrete space. Then X is compact if and only if X is finite.

<u>Proof.</u> If X is finite then from $X = \bigcup U_i$, we can pick $U_{i(x)}$ for $x \in X$ s.t. $x \in U_{i(x)}$ and then $X = \bigcup_{x \in X} U_{i(x)}$ is a finite union.

If X is compact then since $X = \bigcup_{x \in X} \{x\}$ with $\{x\}$ open we can find a finite subset $X' \subset X$ s.t.

$$X = \bigcup_{x \in X'} \{x\} = X'.$$

$\square$

with subspace topology

In particular: if X contains a closed infinite discrete subset, it is not compact

(2) <u>Prop.</u> In $\mathbb{R}^n$, a set $A \subset \mathbb{R}^n$ is compact if and only if A is closed and bounded.

<u>Proof.</u> If A is compact, it is closed (since $\mathbb{R}^n$ is Hausdorff). It is bounded because the ~~identity~~ map $\begin{cases} A \longrightarrow \mathbb{R} \\ x \longmapsto \|x\| \end{cases}$ is locally

bounded, and therefore bounded on the compact set A. (These two necessary conditions hold for *any* metric space, but are *not* sufficient in general.)

Conversely, we consider the case $n=1$ first. Thus let $A \subset \mathbb{R}$ be closed and bounded. Then there exists $a \geq 0$ such that $A \subset [-a, a]$, and since A is closed, it is enough to prove that $[-a, a]$ is compact.

Thus let $(U_i)_{i \in I}$ be an open covering of $[-a, a]$.

Define

$$b = \sup \underbrace{\{ t \in [-a, a] \mid [-a, t] \subset \bigcup_{j \in J} U_j \text{ for some finite set } J \subset I \}}_{\text{not empty because } [-a,-a] = \{a\} \subset U_j \text{ for some } j}$$

We have $b > a$ since $[-a, -a+\varepsilon[$ is contained in some U_j, so that $[-a, -a + \frac{\varepsilon}{2}] \subset U_j$.

We have $b \leq a$, and the next goal is to show that $b = a$. Suppose $b < a$. ~~Then let~~ Pick U_{j_0} such that $b \in U_{j_0}$. Since U_j is open, there is $\varepsilon > 0$ s.t. $]b-\varepsilon, b+\varepsilon[\subset [-a, a]$. By definition of b, there is a $t \geq b - \varepsilon$ s.t.

$$[-a, t] \subset \bigcup_{j \in J} U_j, \quad J \subset I \text{ finite}.$$

But then

$$[-a, b + \tfrac{\varepsilon}{2}] \subset \bigcup_{j \in J} U_j \cup U_{j_0}$$

which contradicts the definition of b. The same argument shows that b is a maximum so we conclude that $[-a, a]$ has a finite subcovering.

In general, for any n we have similarly $A \subset \prod_{i=1}^{n} [-a_i, a_i]$ for some $a_i \geq 0$, so it suffices to prove that these sets are compact. This is a very special case of the compactness

of product spaces (Tychonov's Theorem), which we will see later.
$\square$

(3) Consider the space $L^2([0,1])$ of (equivalence classes of) square-integrable functions $f : [0,1] \longrightarrow \mathbb{C}$ (w.r.t. Lebesgue measure). The set

$$X = \left\{ f \;\middle|\; \|f\|^2 = \int_0^1 |f(x)|^2\,dx \le 1 \right\}$$

is closed and bounded in the metric space $L^2([0,1])$ where $d(f,g) = \left(\int_0^1 |f(t)-g(t)|^2\,dt\right)^{1/2}$. It is not compact: indeed, let

$$A = \left\{ \begin{array}{l} f_n : [0,1] \longrightarrow \mathbb{C} \\ (n\geq 0) \qquad x \longmapsto e^{2i\pi n x} \end{array} \right\} \subset L^2([0,1]).$$

We have $\|f_n - f_m\|^2 = 2$ for all $n \neq m$, and in particular A is discrete with the subspace topology, and also closed in X. (If a sequence from A converges in X, then it is constant from some point on, since the distance ~~from~~ between two terms of the sequence is < 2 from some point on)
So if X were compact, A would be also compact, since $L^2([0,1])$ is Hausdorff, and A would be compact and discrete, it would be finite.

(4) Let $\mathcal{M} = \{\text{measures } \mu \text{ on } [0,1] \text{ s.t. } \mu([0,1]) = 1\}$, and define a topology by U open if and only if, for all $\mu \in U$, there exist $n \geq 1$, $f_1, \dots, f_n$ continuous from $[0,1]$ to $\mathbb{C}$ and $\delta > 0$ s.t.

$$\left\{ \nu \;\middle|\; \left|\int_0^1 f_i\,d\mu - \int_0^1 f_i\,d\nu\right| < \delta \right\} \subset U.$$
$$i = 1, \dots, n$$

Then it is a fundamental fact that $\mathcal{M}$ is compact.

Example (5) – Let X be any topological space and let (x_n) be a sequence in X. Assume it converges to some $y \in X$ (which may not be unique).
Let

$$A = \{x_n \mid n \geq 1\} \cup \{y\}.$$

Then A is compact (with the subspace topology).

Indeed, let $(U_i)_{i \in I}$ be open subsets of X s.t.

$$A \subset \bigcup_{i \in I} U_i.$$

There exists some $i_0 \in I$ s.t. $y \in U_{i_0}$. By definition of convergence, there exists $N \geq 1$ such that

$$n \geq N \Rightarrow x_n \in U_{i_0}$$

(since U_{i_0} is an open neighborhood of y).
We then have indices i_j for $1 \leq j \leq N-1$ s.t.

$$x_j \in U_{i_j}$$

and therefore

$$A \subset \bigcup_{0 \leq j \leq N-1} U_{i_j}.$$

As a concrete example, for $X = \mathbb{R}$, one can take

$$A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\right\} \cup \{0\}$$

as a compact subset of $\mathbb{R}$. (Note that the subset $\left\{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\right\} \subset \mathbb{R}$ is not compact; it is also an example of a discrete set which is not closed, since its closure is A.)

We now discuss the relations between compactness and convergence.

<u>Proposition</u> – Let (X, d) ~~be~~ a metric space. The space X is compact if and only if every sequence (x_n) in X has a convergent subsequence. [$\Leftrightarrow$ X is "sequentially compact"]
(This is not true in general – <u>both</u> directions can fail)

<u>Proof</u> – (1) Assume X is compact. Let (x_n) be a sequence in X and $A = \{x_n \mid n \geq 1\}$ the set of values of (x_n).
We distinguish two cases.
(a) If there exists $x \in A$ s.t. $\{k \mid x_k = x\}$ is infinite, we get a subsequence converging to x.
(b) Otherwise, we claim that if (x_n) has <u>no</u> convergent subsequence, then for all $x \in X$, we can find an open neighb. U_x s.t. $U_x \cap A \subset \{x\}$. Assuming this, we have a covering $X = \bigcup_{x \in X} U_x$, hence a finite set $S \subset X$ s.t. $X = \bigcup_{x \in S} U_x$, and then $A = \bigcup_{x \in S} (U_x \cap A)$ is finite, which is a contradiction.
To check the claim, consider the open balls $B(x, \frac{1}{n})$ for $n \geq 1$. If $B(x, \frac{1}{n}) \cap A \not\subset \{x\}$ for all n, we obtain a subsequence converging to x, (Pick $x_{n_1} \neq x$ in $B(x,1) \cap A$;
~~Then pick n_2 s.t. $\frac{1}{n_2} < 1$ and x_{n_2} in $B(x, \frac{1}{n_2})$~~
~~$d(x, x_{n_1}) > \frac{1}{n_2}$ and $x_{n_2} \neq x$ in $B(x,$~~
Then pick n_2 so that $\frac{1}{n_2} < d(x, x_{n_1})$ and $x_2 \neq x$ in $B(x, \frac{1}{n_2})$, etc...) so we find $U_x = B(x, \frac{1}{N})$ for suitable N with the property we want.

(2) Conversely, assume that every ~~sub~~sequence has a convergent subsequence.

Lemma – (1) If every sequence in X has a convergent subsequence then every $f : X \longrightarrow \mathbb{R}$ which is locally bounded is bounded.
(2) If every locally bounded $f : X \longrightarrow \mathbb{R}$ is bounded, then for any open covering $X = \bigcup U_i$, there exists $\delta > 0$ (called a Lebesgue number) such that for all $x \in X$, there exists $j \in I$ such that $B(x, \delta) \subset U_j$.

Proof – (1) If f is not bounded we find for each $n \geq 1$ some x_n such that $|f(x_n)| \geq n$. Let (x_{n_k}) be a subsequence converging to $x \in X$; then we have a contradiction because $f(x_{n_k})$ must be bounded on the one hand (since f is bounded in a neighborhood of x, in which x_{n_k} lies for k large enough) and yet $|f(x_{n_k})| \geq n_k$.

(2) Define $f : X \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } \underset{\forall r > 0}{\wedge}\ B(x, r) \subset U_j \text{ for some } j \\[2ex] \dfrac{1}{\sup\{\delta > 0 \mid B(x,\delta) \subset U_j \text{ for some } j\}} & \text{otherwise.} \end{cases}$$

We claim that f is locally bounded: indeed, by the triangle inequality, we have $B(y, \frac{\delta}{2}) \subset U_j$ whenever $\begin{cases} d(x,y) < \delta/2 \\ B(x, \delta/2) \subset U_j \end{cases}$ and this implies that

$$f(y) \leq \frac{4}{\delta} \quad \text{for} \quad d(x,y) < \frac{\delta}{2}.$$

By assumption f is bounded, and this gives the result.
$\square$

We now conclude the proof of the Prop. (p°36). Let $X = \bigcup_{j \in J} U_j$ be an open covering of X. Let δ be a Lebesgue number, so any $B(x, \delta)$ is in some U_j. But we claim that there is a <u>finite</u> set $S \subset X$ s.t.

$$X = \bigcup_{x \in S} B(x, \delta)$$

Then since each $B(x, \delta)$ is contained in U_{j_x} for some j_x, we get

$$X = \bigcup_{x \in S} U_{j_x}$$

To prove the claim by contradiction, define inductively a sequence (x_n) by taking $x_1 \in X$ arbitrary (note $X \neq \emptyset$!) then

$x_2 \notin B(x_1, \delta)$ $\quad [X \neq B(x_1, \delta)$ by assumption$]$

$\ldots \quad x_n \notin \bigcup_{1 \leq j \leq n-1} B(x_j, \delta)$.

By construction, we get $d(x_n, x_m) \geq \delta$ if $n \neq m$, and so (x_n) has no convergent subsequence. (Same argument as in Example (3), p°35.) This is a contradiction.

$\square$

To give a "convergence" definition of compactness, one needs the notion of an <u>ultrafilter</u>.

<u>Def.</u> Let X be a set. An <u>ultrafilter</u> $\mathcal{F}$ on X is a filter such that for <u>any</u> $A \subset X$, either $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$.

<u>Example.</u> Let $x \in X$. Then

$$\mathcal{F}_x = \{ A \subset X \mid x \in A \}$$

is an ultrafilter on X. Such ultrafilters are called <u>principal</u>. They are basically the <u>only</u> ultrafilters that can be explicitly described. Yet...

Lemma. Let $\mathcal{F}_0$ be a filter on a set X. Then there is at least one ultrafilter which contains $\mathcal{F}_0$.

$\neq \emptyset$ since $\mathcal{F}_0 \in \mathcal{O}$

Proof. This follows from Zorn's Lemma: let $\mathcal{O}$ be the set of filters containing $\mathcal{F}_0$; order $\mathcal{O}$ by inclusion and note that if $Q \subset \mathcal{O}$ is totally ordered, then it has an upper bound: let

$$\mathcal{F} = \{A \mid \exists \mathcal{G} \in Q, \ A \in \mathcal{G}\} = \bigcup_{\mathcal{G} \in Q} \mathcal{G}.$$

Then $\mathcal{F}$ is a filter, $\mathcal{F}$ contains $\mathcal{F}_0$, and any element of Q is contained in $\mathcal{F}$. (Condition (2) of the definition of filters is the only slightly tricky one: if A_1, A_2 are in $\mathcal{F}$, then $A_i \in \mathcal{G}_i$ for some $\mathcal{G}_i \in Q$; then either $\mathcal{G}_1 \subset \mathcal{G}_2$ and $A_1 \cap A_2 \in \mathcal{G}_2$ or $\mathcal{G}_2 \subset \mathcal{G}_1$ and $A_1 \cap A_2 \in \mathcal{G}_2$; in any case $A_1 \cap A_2 \in \mathcal{F}$.)

So Zorn's Lemma gives a maximal element of $\mathcal{O}$; we claim such an element $\mathcal{F}$ (which contains $\mathcal{F}_0$) is an ultrafilter. Indeed, let $A \subset X$ be any set. If $A \notin \mathcal{F}$ and $X - A \notin \mathcal{F}$ then

$$\mathcal{F}' = \{B \subset X \mid A \cup B \in \mathcal{F}\}$$

is a filter on X (check!), contains $\mathcal{F}_0$, and contains $X - A$, so $\mathcal{F}' \supsetneq \mathcal{F}$, contradicting the maximality of $\mathcal{F}$. $\square$

Proposition. A topological space X is compact if and only if every ultrafilter on X converges.

Proof. $\Leftarrow$: assume that every ultrafilter converges; let

$$X = \bigcup_{i \in I} U_i$$

be an open covering of X. Then $\emptyset = \bigcap_{i \in I} C_i$, with $C_i = X - U_i$, closed.

If there is no finite subcovering, then we claim that

$$\mathcal{F} = \{ A \subset X \mid \exists J \subset I \text{ finite}, A \supset \bigcap_{j \in J} C_j \}$$

is a <u>filter</u> on X.

(Indeed, $\emptyset \notin \mathcal{F}$ as this would give $\bigcap_{j \in J} C_j = \emptyset$ for some J finite, so $X = \bigcup_{j \in J} U_j$; the other conditions are easy.)

By the Lemma (p° 39), there is an ultrafilter $\tilde{\mathcal{F}}$ containing $\mathcal{F}$. By assumption, $\tilde{\mathcal{F}}$ converges to some $x_0 \in X$. We then get a contradittion because such an x_0 is in $\bigcap_{i \in I} C_i$: for any $i \in I$, and any neighborhood U of x_0, we have

$$C_i \in \mathcal{F} \subset \tilde{\mathcal{F}} \quad \text{(by definition)}$$

$$U \in \tilde{\mathcal{F}} \quad \text{(by ——— of convergence)}$$

so $C_i \cap U \in \tilde{\mathcal{F}}$, and in particular $C_i \cap U \neq \emptyset$. This means that $x_0 \in \overline{C_i} = C_i$.

<u>$\Rightarrow$</u>: assume that X is compact, and let $\mathcal{F}$ be an ultrafilter on X. We claim that if $\mathcal{F}$ does <u>not</u> converge, then

$$\bigcap_{A \in \mathcal{F}} \bar{A} = \emptyset$$

but any <u>finite</u> intersection is non-empty, contradicting compact-~~ne~~-ness for $X = \bigcup_{A \in \mathcal{F}} (X - \bar{A})$.

Indeed, first ~~...~~ any finite intersection of $\bar{A}$ is an element of $\mathcal{F}$, so is not empty.

Second, if $\bigcap_{A \in \mathcal{F}} \bar{A}$ contains some x_0, then $\mathcal{F}$ would converge to x_0, contrary to the assumption (let U be an open neighb. of x_0; we have $U \in \mathcal{F}$ because otherwise $X - U \in \mathcal{F}$, and then $X - U$ is a closed set in $\mathcal{F}$ so $x_0 \in X - U$, which is a contradiction).

<u>Note:</u> An ultrafilter $\mathcal{F}$ on X is principal $\Leftrightarrow$ there is a <u>finite</u> set in $\mathcal{F}$.