3. Completeness [Ref. Bourbaki, "General Topology", Ch. 2]

The notion of Cauchy sequence in $\mathbb{R}$ has an immediate extension to metric spaces: (x_n) is a Cauchy sequence if and only if

$$\forall \varepsilon > 0, \ \exists N, \ \forall n, m \geq N, \ d(x_n, x_m) < \varepsilon.$$

However, because this involves comparing two arbitrary points of X, this cannot be generalized in an arbitrary topological space (it is possible to say that "y is close to x", for given x, using neighborhoods of x, but we should somehow have "uniform" control on all neighborhoods of all x).

There is a way around that...

Def. Let X be a set. A uniform structure on X is a collection $\mathcal{U}$ of subsets of $X \times X$ s.t.

(1) $A \in \mathcal{U}$ and $B \supset A \Rightarrow B \in \mathcal{U}$

(2) A_1, A_2 in $\mathcal{U} \Rightarrow A_1 \cap A_2 \in \mathcal{U}$

(3) $\Delta = \{(x,x) \mid x \in X\} \subset A$ for all $A \in \mathcal{U}$

(4) If $A \in \mathcal{U}$, then $A^{-1} = \{(x,y) \mid (y,x) \in A\} \in \mathcal{U}$

(5) If $A \in \mathcal{U}$, there exists $B \in \mathcal{U}$ such that
$$\{(x,y) \mid \exists z, (x,z) \in B, (z,y) \in B\} \subset A.$$

Ex. (1) (X, d) metric space
$$\mathcal{U} = \{A \subset X \times X \mid \exists \delta > 0, \ d(x,y) < \delta \Rightarrow (x,y) \in A\}$$
is a uniform structure on X.

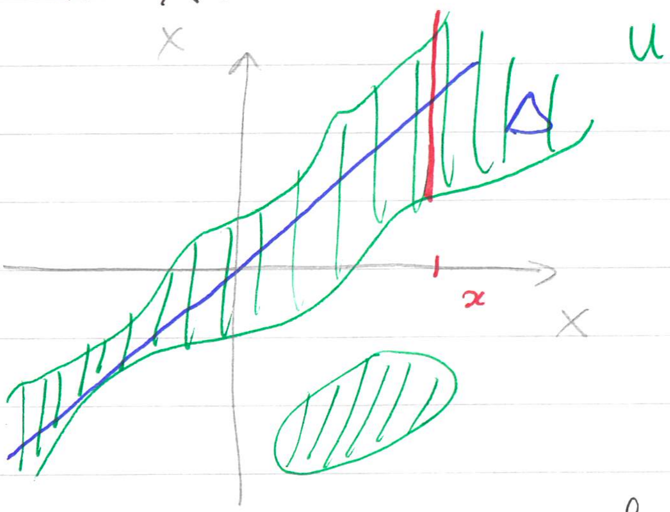
(2) X topological group: $\mathcal{U} = \{A \subset X \times X \mid \exists U$ neigh. of 1 s.t. $\{(x,y) \mid xy^{-1} \in U\} \subset A\}$

~~(3) X compact space~~

is a uniform structure.

(3) Let X be a compact ~~space~~ Hausdorff space. / for the product topology

$\mathcal{U} = \{ A \subset X \times X \mid \exists\, U \subset X \times X$ open s.t. $U \subset A \}$ (neighborhoods of the diagonal) is a uniform structure on X.



Given $\mathcal{U}$ on X, uniform structure, there is an associated topology: a neighborhood of $x \in X$ is a set of the form $V = \{ y \in X \mid (x,y) \in U \}$ for some $U \in \mathcal{U}$ (check that this is a topology!)

Ex. In cases of metric spaces and compact spaces, we recover the topology we started with from the uniform structure of the previous example.

Now the key fact is that one can define Cauchy filters: a filter $\mathcal{F}$ on X is a Cauchy filter $\Leftrightarrow$ for all ~~neigh.~~ $U \in \mathcal{U}$, there exists a ~~neighb~~ ~~of~~ set $A \subset X$ s.t. $A \times A \subset U$ and $A \in \mathcal{F}$.

Ex. If $\mathcal{F}$ is the elementary filter associated to (x_n), then this gives Cauchy sequences in metric spaces.

Def. X with a uniform structure is complete $\Leftrightarrow$ every Cauchy filter converges.

Then we have an analogue of completion:

__Theorem__ - If $(X, \mathcal{U})$ is a uniform space, there exists a continuous map $i: X \longrightarrow \hat{X}$ with $\hat{X}$ $\left\{\begin{array}{l}\text{Hausdorff and}\\ \text{complete}\end{array}\right.$, s.t. $i(X)$ is __dense__ and $i: X \longrightarrow i(X)$ is a homeomorphism. This is unique in the sense that if $j: X \longrightarrow \hat{X}'$ is another, ~~then~~ there is a unique homeomorphism $f: \hat{X} \longrightarrow \hat{X}'$ s.t. $f \circ i = j$.

__Ex.__ $\mathbb{Q} \xrightarrow{i} \mathbb{R}$; $\mathbb{R}^n$ is complete and Hausdorff for $n \geq 0$

__Theorem__ - If X is __compact__ and Hausdorff then:

(1) the uniform structure of Example (3) is the __unique__ uniform structure whose associated topology is that of X.

(2) X is __complete__.

__Proposition__ - (1) If (X, d) is a metric space, it is complete if and only if all Cauchy sequences converge.

(2) If X is $\left(\begin{array}{l}\text{Hausdorff}\\ \text{complete}\end{array}\right.$, then $Y \subset X$ is complete if and only if Y is closed. [The uniform structure on Y is $\{ W \cap (Y \times Y) \mid W \in \mathcal{U} \}$]

Uniform structures also provide the generalization of __uniformly continuous__ functions: $f: \underset{\mathcal{U}}{X} \rightarrow \underset{\mathcal{V}}{Y}$ is uniformly continuous if, for all $V \in \mathcal{V}$, there is a $U \in \mathcal{U}$ s.t.

$$(x, y) \in U \implies (f(x), f(y)) \in V$$

For metric spaces, we get the usual uniform continuity.

__Prop.__ X [redacted] compact; any $f: X \rightarrow Y$ is uniformly continuous.