2 - Connectedness

We now consider the second property of topological spaces mentioned at the beginning of this chapter: connectedness. Intuitively, this means that X is "in one piece". Another intuitive point of view is that "X has no non-trivial discrete invariants". This can be taken as the definition.

Definition. Let X be a topological space. The space X is connected if any continuous map $f: X \to D$, where D is discrete, is constant.

This looks like it may be difficult to check. However, not all D need be considered.

Lemma. (1) X is connected $\iff$ any discrete $f: X \to \{0,1\}$ continuous is constant.

(2) X is ~~constant~~ connected $\iff$ whenever $X = U_1 \cup U_2$ with U_i open and $U_1 \cap U_2 = \emptyset$, we have $U_1 = X$ or $U_2 = X$.

(3) X is connected $\iff$ whenever $X = C_1 \cup C_2$ with C_i closed and $C_1 \cap C_2 = \emptyset$, we have $C_1 = X$ or $C_2 = X$.

(4) X is connected $\iff$ $\emptyset$ and X are the only subsets of X which are both open and closed.

with $\begin{cases} U_i = X - C_i \\ C_i = X - U_i \end{cases}$

Proof. (2) $\iff$ (3) $\iff$ (4) is easy: (2) $\iff$ (3) ~~by replacing U_i with~~ and (2) $\iff$ (4) because if U is open and closed then

$$X = U \cup (X - U)$$

is a union of two disjoint open sets.

(1) $\iff$ (2) because if $X = U_1 \cup U_2$, we can define $f(x) = \begin{cases} 1, & x \in U_1 \\ 0, & x \in U_2 \end{cases}$

↳ with U_i open, disjoint

nd check that it is continuous $\left(f^{-1}(\{0\}) = U_2,\ f^{-1}(\{1\}) = U_1 \right)$, and constant if and only if <u>either</u> $U_1 = X$ or $U_2 = X$.

Let us prove (1) then. It suffices to show $\Leftarrow$.

Thus let $f : X \to D$ ~~be~~ continuous with D discrete. By replacing D by the image of f, we can assume f surjective.

If D has then 1 element, then f is constant.

Suppose D has ≥ 2 elements. Pick $i_0 \in D$.

Then define $g : X \longrightarrow \{0, 1\}$ by

$$g(x) = \begin{cases} 1 & \text{if } f(x) = i_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then g is continuous $\Big(g^{-1}(\{1\}) = f^{-1}(\underbrace{\{i_0\}}_{\text{open in } D})$ and

$$g^{-1}(\{0\}) = f^{-1}(\underbrace{D - \{i_0\}}_{\text{open in } D}) \Big)$$

so constant. But we assumed f is surjective, so both $g^{-1}(\{1\})$ and $g^{-1}(\{0\})$ are not empty, and this is a contradiction.

$\square$

<u>Examples.</u> (1) A discrete space is connected if and only if it has ≤ 1 elements (in particular, $\emptyset$ is connected — note that some books exclude this!)

(2) <u>Prop.</u> - In $\mathbb{R}$, a set $A \subset \mathbb{R}$ is connected (with the subspace topology) $\Leftrightarrow$ A is an <u>interval</u>.

<u>Proof.</u> $\Rightarrow$: an interval is a set $A \subset \mathbb{R}$ s.t.

$$(a \leq c \leq b,\ a, b \in A) \Rightarrow c \in A.$$

If A is <u>connected</u> and $a < c < b$ with

a and b in A then we must have also $c \in A$ since otherwise

$$A = \underbrace{(]-\infty, c[\cap A)}_{\text{open in } A} \cup \underbrace{(]c, +\infty[\cap A)}_{\text{open in } A}$$

would contradict the connectedness of A. This handles the case when A has ≥ 2 elts.
If A is $\emptyset$ or $\{a\}$, then it is also an interval.

$\underline{\Leftarrow}$: Consider first $A = [a, b]$ with $a \leq b$. Let $A = U_1 \cup U_2$ with $\begin{cases} U_i \subset A \text{ open} \\ U_1 \cap U_2 = \emptyset \end{cases}$. Assume that $a \in U_1$; we need to show that $A = U_1$.

Let $c = \operatorname{Sup} \{ t \in [a, b] \mid [a, t] \subset U_1 \}$.
We claim first that $[a, c] \subset U_1$. Indeed, otherwise $c \in U_2$ (if $c \in U_1$, then since $[c, c-\varepsilon]$ contains an element t with $[a, t] \subset U_1$, for all $\varepsilon > 0$, we would get $[a, c[\subset U_1$ and then $[a, c] \subset U_1$), so for some $\varepsilon > 0$, $]c - \varepsilon, c + \varepsilon[\cap A$ is contained in U_2, which would give $c \leq c - \varepsilon$, a contradiction.

Now we claim that $c = b$ to conclude the proof: otherwise we find $\varepsilon > 0$ s.t. $]c - \varepsilon, c + \varepsilon[\cap A \subset U_1$ and $c + \varepsilon < b$, so $[a, c + \frac{\varepsilon}{2}] \subset U_1$, which is a contradiction to the definition of c.

To handle a general interval, we use a general useful lemma.

<u>Lemma</u>. Let X be a topological space. Let $(A_i)_{i \in I}$ be any family of subsets of X, connected for their subspace topologies. If $\bigcap_{i \in I} A_i \neq \emptyset$, then $\bigcup_{i \in I} A_i$ is connected.

Proof. Let $f : \bigcup A_i \to \{0,1\}$ be continuous. Then f restricted to A_i is continuous for each i, so constant. But then for $x_0 \in \bigcap A_i$, we get $f(x) = f(x_0)$ for all $x \in A_i$ for all i, so f is constant.

□

Using this and the first case we get:

$$[a, b[\; = \bigcup_{\substack{n \geq 1 \\ b - 1/n \geq a}} \underbrace{[a, b - \tfrac{1}{n}]}_{\text{all containing } a} \quad \text{connected}$$

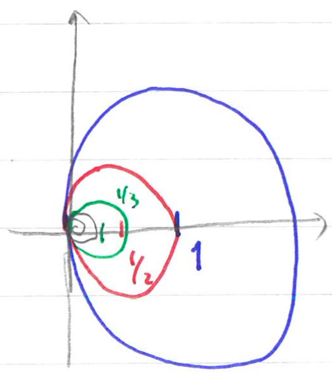
$$[a, +\infty[\; = \bigcup_{n \geq 1} \underbrace{[a, a+n]}_{\text{all containing } a} \quad \text{connected}$$

$$\mathbb{R} = \bigcup \underbrace{[-n, n]}_{\text{all containing } 0} \quad \text{connected}$$

etc...

□

Example - (3) Let $X \subset \mathbb{R}^2$ be the union of the circles C_n of radius $\frac{1}{n}$, $n \geq 1$, centered at $\frac{1}{n}$:



Since $0 \in C_n$ for all n, and each C_n is connected (see Lemma below) we deduce that X is connected. (This space has many interesting properties.)

We used one of the basic tools to check that circles in $\mathbb{R}^2$ are connected, namely:

Lemma. Let $f: X \to Y$ be continuous. If X is connected, then $f(X) \subset Y$ is connected. (But $f^{-1}(Y)$ is rarely connected even if Y is, for example for f constant, and X not connected.)

Proof. Let $g : f(X) \longrightarrow D$ be continuous with D discrete. Then $g \circ f : X \longrightarrow D$ is continuous, so it is constant, so g is constant on $f(X)$.
$\square$

~~The~~ Taking $f(t) = re^{2i\pi t} + c$ for $t \in [0,1]$, it follows that any circle is connected in $\mathbb{R}^2$.

~~Connectedness is a useful property, like compactness, but~~

Corollary. (Intermediate value theorem)
If $f: \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and $f(a) < c < f(b)$, there is an x between a and b s.t. $f(x) = c$.

(Indeed, if $a \leq b$, $f([a,b])$ is an interval (since it is connected) containing $f(a)$ and $f(b)$.)

In general, connectedness is a very useful property to have, like compactness. But a difference is that one can often reduce a problem to a connected case, by looking at connected components.

Definition. Let X be a topological space. Let $x_0 \in X$. The connected component of x_0 in X is the union of all connected subspaces of X containing x_0.

By the lemma on p° 43, the connected component of x_0 is a (non-empty) connected subspace of X; it is the largest connected set containing x_0.

It is possible that the connected component of x_0 is $\{x_0\}$. If this is true for <u>all</u> x_0, the space X is called <u>totally disconnected</u>. Examples include discrete spaces, but there are many others, for instance $\mathbb{Q} \subset \mathbb{R}$.

<u>Proposition</u>. (1) For all $x_0 \in X$, the connected component of x_0 is closed.

(2) The connected components of the elements of X form a partition of X, which are equivalence classes for the relation "$x \sim y \iff x$ is in the connected component of y."

<u>Proof.</u> (1) The point is that if A is connected, so is $\bar{A}$. Indeed, if $\bar{A} = U_1 \cup U_2$ with $U_i \neq \emptyset$, $U_1 \cap U_2 = \emptyset$, U_i open, then $U_i \cap A$ is not empty (by def. of closure), and $A = (A \cap U_1) \cup (A \cap U_2)$ gives a contradiction to the connectedness of A.

(2) If x is in the connected component V_y of y, then V_y is a connected set with $x \in V_y$, so $V_y \subset V_x$. But then $V_x = V_y$ by definition, so the relation in question is just "$V_x = V_y$", which is an equivalence relation. The equivalence class of x is exactly V_x.

$\square$