## 4 - Local versions

In general, any topological notion can be "localized" by asking that points have neighborhoods which satisfy the condition.

**Definition** - Let $X$ be a topological space.

(1) $X$ is locally compact if every $x \in X$ has a fundamental system of compact neighborhoods.

(2) $X$ is locally connected if every $x \in X$ has a ______ ______ connected ______.

[ (3) A metric space is locally complete if every $x \in X$ has a fundamental system of complete neighborhoods.]

Warning: locally closed means something different than one might then expect!

These have in fact very different properties.

**Examples** - (1) Any discrete space is locally compact and locally connected.

(2) $\mathbb{R}^n$, for $n \geq 1$, is locally compact and locally connected, since the closed balls $B(x, \delta)$ with $\delta > 0$ form a system of compact / connected neighborhoods of $x$.

(3) $L^2([0,1])$ is not locally compact, but is locally connected.

(4) Any topological manifold is locally compact and locally connected.

**Proposition** – Let $X$ be a Hausdorff topological space. The space $X$ is locally compact if and only if every $x \in X$ has one compact neighborhood.

We need another useful lemma:

**Lemma** – If $X$ is a compact Hausdorff space, then every $x \in X$ has a f.s. of closed neighborhoods.

**Proof.** Let $x \in X$ and $U \subset X$ an open neighb. of $x$. Let $C = X - U$ which is closed. Since $x \notin C$, $C$ closed, $X$ compact Hausdorff, we can find $\left.\begin{array}{l}\text{an open neighb. } V \text{ of } x \\ \text{an open set } W \supset C\end{array}\right\}$ such that $V \cap W = \emptyset$ (cf. p. 32)

Then $\overline{V} \cap W = \emptyset$: indeed, if $y \in W$, there is an open neigh. $V_y$ of $y$ in $W$, so that $V_y \cap V = \emptyset$, so $y \notin \overline{V}$.

It follows that $\overline{V} \subset U$, and is a closed neighb. of $x$

$\square$

**Proof of Prop.** – Let $x \in X$ and let $C \subset X$ be a compact neighborhood of $x$. Let $U$ be an open neighb. of $x$. Since $U \cap C$ is an open neighb. of $x$ in $C$, the lemma (applied to $C$) gives a closed neighb. $C'$ of $x$ in $C$ s.t. $C' \subset U \cap C$.

Then: (1) $C' \subset U$ (by construction)

(2) $C'$ is compact in $X$ (indeed, $C'$ is closed, hence compact in $C$, which is compact Hausdorff, then $C' = i(C')$ where $i : C \longrightarrow X$ is the inclusion, so $C'$ is compact in $X$).

(3) $C'$ is a neighborhood of $x$ in $X$:

let $W \subset C$ be an open neighborhood of $x$, and let $V' \subset C'$ be an open neighborhood of $x$ in $C$; we have $V' = C \cap V''$ for some open neighborhood of $x$ in $X$, and

$$C' \supset V' = C \cap V'' \supset V'' \cap W$$

which is an open neighborhood of $x$ of $X$.)

As a corollary, a compact Hausdorff space is locally compact. On the other hand, "locally connected" and "connected" are <u>disjoint</u> notions (see exercises).

<u>Proposition</u>. Let $X$ be a topological space.
$X$ is locally connected $\iff$ for all $U \subset X$ and all $x \in U$, the connected component of $x$ <u>in $U$</u> is open (in $X$, or equivalently in $U$).
In particular, if $X$ is locally connected, then the connected components of elements of $X$ are open and closed.

<u>Proof</u>. $\Rightarrow$: let $U \subset X$ be open, $x \in U$, $V$ the connected component of $x$ in $U$; let $y \in V$ and $W$ a connected neighb. of $y$ contained in $U$, which exists by assumption; then $W$ is contained in the component of $y$ in $U$, which is the same as that of $x$ (cf. p° 46); so $W \subset V$. Hence $V$ contains neighb of each of its points, so it is open.

$\Leftarrow$: let $x \in X$ and $U$ an open neighb. of $x$; let $V$ be the connected component of $x$ in $U$; by assumption, $V$ is open, so is a connected neighb. of $x$ contained in $U$.
$\square$