3 – Functions

Recall we denote by $\mathcal{C}(X)$ the set of continuous functions from X to $\mathbb{C}$. This is in fact a vector space, and even a ring, since the sum/product of continuous functions remain so: for instance $fg = m \circ (f,g)$ where

$$\begin{array}{ccccc} (f,g): & X & \longrightarrow & \mathbb{C}\times\mathbb{C} & \xrightarrow{\ m\ } & \mathbb{C} \\ & x & \longmapsto & (f(x), g(x)) & & \\ & & & (a,b) & \longmapsto & ab \end{array}$$

and (f,g) is continuous when $\mathbb{C}\times\mathbb{C}$ has the product (or equivalently euclidean) topology, as is $m : \mathbb{C}^2 \longrightarrow \mathbb{C}$.

Also, the constant functions are elements of $\mathcal{C}(X)$.

However, in general it is not clear at all that $\mathcal{C}(X)$ contains any non-constant function!

Ex. If (X,d) is a metric space, then $d : X\times X \longrightarrow \mathbb{R}$ is continuous (for the product topology on $X\times X$), so we get plenty of function of the type

$$f(x) = \sum_{j\in J} c_j \, d(x_j, x)$$

with J finite, $c_j \in \mathbb{C}$, $x_j \in X$.

Remarkably, very simple conditions imply the existence of many continuous functions on a topological space.

Theorem (Urysohn, 1924) – Let X be a Hausdorff topological space. The following are equivalent:

(1) If A, B are closed subsets of X with $A\cap B = \emptyset$, there is $f : X \to [0,1]$ continuous s.t. $A \subset f^{-1}(0)$, $B \subset f^{-1}(1)$.

(Tietze extension theorem)

(2) If $A \subset X$ is closed and $g : A \longrightarrow I$ is continuous, where $I = [a,b]$, $a \leq b$, then there is a $f : X \longrightarrow I$ continuous extending g.

(3) If A, B are closed subsets of X with $A \cap B = \emptyset$, there exist open sets $U \supset A$ and $V \supset B$ such that $U \cap V = \emptyset$.

(3bis) If $A \subset U \subset X$ with A closed and U open, there exists $V \subset X$ open such that $A \subset V \subset \overline{V} \subset U$.

__Def.__ A Hausdorff space X satisfying these conditions is called normal.

__Sketch of the proof.__ (3) $\iff$ (3bis) : take $B = X - U$, etc.

(1) $\Rightarrow$ (3) : given f with the property of (1), take

$$U = f^{-1}([0, \tfrac{1}{3}[), \quad V = f^{-1}(]\tfrac{2}{3}, 1])$$

for instance.

(2) $\Rightarrow$ (1) : given A, B with $A \cap B = \emptyset$, let $C = A \cup B$; then C is closed and

$$g : C \longrightarrow [0,1]$$
$$x \longmapsto \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$$

is continuous. Extending it to X gives the desired function $f : X \longrightarrow [0,1]$.

It would then be enough to prove (3bis) $\Rightarrow$ (2), but in fact we will show that (3bis) $\Rightarrow$ (1) $\Rightarrow$ (2).

(3bis) $\Rightarrow$ (1):

__Claim__ : There exist open sets U_t, $t \in [0,1]$, such that

(1) $A \subset U_0$

(2) $B \subset X - U_1$

(3) $s < t \Rightarrow \overline{U_s} \subset U_t$

If we assume this, we define $f : X \longrightarrow [0,1]$ by

$$\begin{cases} f(x) = 1 , & x \notin U_1 \\ f(x) = \inf \{ s \in [0,1] \mid x \in U_s \} , & x \in U_1 \end{cases}$$

Then $f(A) = \{0\}$ by (1), $f(B) = \{1\}$ by (2), and moreover f is <u>continuous</u>: indeed, let $x \in X$ and $a = f(x) \in [0,1]$. Then for $\varepsilon > 0$ we get

$$a - \varepsilon \leq f(y) \leq a + \varepsilon$$

if $y \in U_{a+\varepsilon} - \overline{U}_{a-\varepsilon}$, and $U_{a+\varepsilon} - \overline{U_{a-\varepsilon}}$ is an open set containing x, so is a neighb. of x.

To check the claim: we first define U_t for $t = \frac{i}{2^n}$, $n \geq 0$, $0 \leq i \leq 2^n$ ("dyadic numbers"), by induction on n:

$U_1 = X - B$

U_0 is an open set s.t. $A \subset \overline{U}_0 \subset U_1$ (exists by (3bis)!)

$U_{1/2}$ is an open set s.t. $\overline{U}_0 \subset U_{1/2} \subset \overline{U_{1/2}} \subset U_1$

$\vdots$

This gives $U_s \subset U_t$ for $s \leq t$ both dyadic. Then let

(*) $$U_t = \bigcup_{\substack{s \leq t \\ s \text{ dyadic}}} U_s , \qquad t \in [0,1].$$

This has the expected properties: indeed, if $t < t'$, we can find dyadic numbers $s < s'$ s.t. $t \leq s < s' \leq t'$, hence

$$\overbrace{\overline{U}_t \subset \overline{U}_s}^{\text{by } (*)} \subset \overbrace{U_{s'} \subset U_{t'}}^{} .$$
$$\underbrace{\overline{U}_s \subset U_{s'}}_{\text{by the dyadic case}}$$

(1) $\Rightarrow$ (2): consider the interval $I = [-1, 1]$.

<u>Lemma</u>. If $A \subset X$ is closed and $f : A \longrightarrow [-1,1]$ continuous

there exists $g : X \longrightarrow [-\frac{1}{3}, \frac{1}{3}]$, continuous, s.t. $|f-g| \leq \frac{2}{3}$ on A.

<u>Proof.</u> Let $B = f^{-1}([-1, -\frac{1}{3}])$, $C = f^{-1}([\frac{1}{3}, 1])$; then B, C are closed and disjoint in X; by (1) we find

$$g_0 : X \longrightarrow [0, 1]$$

continuous with $g_0(B) = \{0\}$, $g_0(C) = \{1\}$. Then we define

$g = \frac{2}{3} g_0 - \frac{1}{3}$, so that

$$g(B) = \{-\tfrac{1}{3}\}, \; g(C) = \{\tfrac{1}{3}\}.$$

We then see that

$$|f - g| \leq \frac{2}{3}$$

on all of A.

□

Now by induction, we construct $(g_n)_{n \geq 0}$ in $\mathscr{C}(X)$ with

$$g_n : X \longrightarrow [-\tfrac{1}{3}, \tfrac{1}{3}]$$

and

on A, $\quad \left| f - g_0 - \frac{2}{3} g_1 - \cdots - \left(\frac{2}{3}\right)^k g_k \right| \leq \left(\frac{2}{3}\right)^k.$

The series $\sum_{i=0}^{+\infty} g_i(x) \left(\frac{2}{3}\right)^i$ is then convergent for all x and defines a function $g : X \longrightarrow \mathbb{C}$, which coincides with f on A, and which converges <u>uniformly</u> on X, and is therefore continuous.

□

<u>Examples.</u> (1) Any metric space satisfies the conditions of the Theorem, hence is normal (e.g. define

$$f_A(x) = \inf_{a \in A} d(x, a), \quad f_B(x) = \inf_{b \in B} d(x, b)$$

then note that f_A, f_B are continuous, because e.g.

$$|f_A(x) - f_A(y)| \leq d(x,y)$$

for $(x,y) \in X \times X$ and that $f_A(x) = 0$ if $x \in A$, $f_B(x) = 0$ if $x \in B$; then

$$U = \{x \mid f_A(x) < f_B(x)\}$$
$$V = \{x \mid f_B(x) < f_A(x)\}$$

are disjoint open subsets of X with

$$A \subset U, \ B \subset V. \ )$$

(2) Any <u>compact Hausdorff</u> space X is normal: for any $x \in A$, find U_x open neighb. of x and V_x open, $V_x \supset B$, s.t. $U_x \cap V_x = \emptyset$, then $A \subseteq \bigcup U_x$, but A is compact (since it is closed in a compact Hausdorff space), so there is a finite subset $Y \subset A$ s.t.

$$A \subset \bigcup_{y \in Y} U_y.$$

Then the open sets

$$\begin{cases} U = \bigcup_{y \in Y} U_y \\ V = \bigcap_{y \in Y} V_y \end{cases}$$

are disjoint, $U \supset A$ and $V \supset B$.

(3) One can check that the space $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$ with the topology of <u>pointwise</u> convergence is <u>not</u> normal (in other words $\mathbb{R}^{\mathbb{R}}$, or similarly $\mathbb{C}^{\mathbb{R}}$, is not a normal space).

There are two further important results concerning continuous functions: the Ascoli-Arzela Theorem (characterizing compact subsets of the space $\mathcal{C}(X)$) and the Stone-Weierstrass Theorem (giving conditions ensuring that a subset of the space of continuous functions is dense for the topology of uniform convergence).

Def. Let X, Y be topological spaces, $\mathcal{C}(X,Y)$ the set of continuous maps $X \to Y$. The compact-open topology is the topology with a basis given by

$$\{ f : X \to Y \mid \forall j \in J,\ f(C_j) \subset U_j \}$$

where J is finite, $C_j \subset X$ is compact and $U_j \subset Y$ is open. One also says, at least if Y is a metric space, that this is the topology of uniform convergence on compact subsets

$Y = \mathbb{C}$

Ex. $f_n \to f$ for the compact open topology if and only if for all $C \subset X$ compact, $f_n \to f$ uniformly on C:

$\forall C \subset X$ compact, $\forall \varepsilon > 0, \exists N, \forall n \geq N, \forall x \in C, |f_n(x) - f(x)| < \varepsilon$

(This is not completely obvious)

Th. (Ascoli) - Let X be a locally compact Hausdorff space and (Y, d) a metric space. A subset F of $\mathcal{C}(X, Y)$, with the compact open topology, is relatively compact ($\Leftrightarrow \overline{F}$ is compact) if and only if

(1) $\forall x \in X$, $\{ f(x) \mid f \in F \} \subset Y$ is ~~compact~~ relatively compact

(2) F is equicontinuous:

$\forall x \in X, \forall \varepsilon > 0, \underbrace{\forall f \in F}_{\exists U \text{ neigh. of } x}\ y \in U \Rightarrow d(f(x), f(y)) < \varepsilon$

the same U works for all $f \in F$

Sketch of the proof (of sufficiency): for all $x \in X$, let
$$C_x = \{ f(x) \mid f \in \overline{\mathcal{F}} \}$$
which is ~~closed~~ compact in Y.

Let
$$C = \prod_{x \in X} C_x$$
which is compact with the product topology (Tychonov).

The map
$$\varphi \begin{cases} \overline{\mathcal{F}} \longrightarrow C \\ f \longmapsto (f(x))_{x \in X} \end{cases}$$
is injective.

We claim that

① $\varphi(\overline{F})$ is closed, hence compact.

② φ defines an homeomorphism $\overline{F} \longrightarrow \varphi(\overline{F})$

This implies the result.

We explain ① and ② when $Y = \mathbb{C}$ (or a metric space) and assuming we can use sequences, to make things more transparent.

① We need to show that if $\varphi(f_n)$ converges to $(g(x))_{x \in X}$ in C, then $g : x \longmapsto g(x)$ is in $\overline{F}$, which means in particular that g is continuous.

The convergence in C is pointwise convergence. This doesn't imply continuity of the limit in general, but this ~~follows~~ comes from equicontinuity as follows: for x, y in X, we have for any $n \geq 1$
$$|g(x) - g(y)| \leq |g(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - g(y)|.$$
Let $\varepsilon > 0$ be given. By equicontinuity, there is a neighb. U of x s.t. $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$ if $n \geq 1$, $y \in U$ (since all f_n are in $\overline{F}$, and it is elementary that $\overline{F}$ is also equicontinuous).

Now let $y \in U$; pick $n \geq 1$ s.t. $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ and $|f_n(y) - f(y)| < \frac{\varepsilon}{3}$ (pointwise convergence); then we get

$$\forall y \in U, \quad |g(x) - g(y)| < \varepsilon.$$

This shows that g is continuous.

(2) It suffices to show that if (f_n) is a sequence in $\overline{F}$ which converges pointwise (to f), then it <u>converges uniformly</u> on all compact subsets of X. (This also shows that the g in (1) is in $\overline{F}$).

Again this fails in general, but comes from equicontinuity: for all $x \in X$, all $\varepsilon > 0$, there is $N(x, \varepsilon) \geq 1$ s.t. $N(x, \epsilon)$ is minimal with

$$n \geq N(x, \varepsilon) \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

<u>Claim</u>: for given $\varepsilon > 0$, the function $x \longmapsto N(x, \varepsilon)$ is locally bounded.

It follows that $x \longmapsto N(x, \varepsilon)$ is bounded on any compac subset $A \subset X$, which proves the uniform converge.

To check the claim: for x, y in X

$$|f_n(y) - f(y)| \leq |f_n(y) - f_n(x)| + |f_n(x) - f(x)| + |f(x) - f(y)|$$

Using equicontinuity of $\{f_n\} \cup \{f\}$, we find a neighb. U_x of x s.t. if $y \in U_x$, we have

$$|f_n(y) - f_n(x)| < \frac{\varepsilon}{3}, \quad |f(y) - f(x)| < \frac{\varepsilon}{2}$$

so that

$$|f_n(y) - f(y)| \leq \frac{2\varepsilon}{3} + |f_n(x) - f(x)|$$

and hence

$$N(y, \varepsilon) \leq N\left(x, \frac{\varepsilon}{3}\right)$$

for $y \in U_x$.

$\square$

Finally, we recall the classical theorem of Weierstrass:

<u>Th.</u> If $[a,b]$ is a compact interval in $\mathbb{R}$, then any $f \in \mathcal{C}([a,b])$ is the uniform limit of a sequence (f_n) of <u>polynomials</u> (restricted to $[a,b]$).

The following is an important generalization:

<u>Th.</u> (Stone - Weierstrass Th.) - Let X be a compact Hausdorff space, $F \subset \mathcal{C}(X, \mathbb{R})$ a ~~space~~ set a real-valued continuous functions on X.
If F is stable by addition and multiplication, contains the constant functions and <u>separates the points</u> of X (i.e. $\forall x \neq y$ in X, there is $f \in F$ s.t. $f(x) \neq f(y)$) then F is dense in $\mathcal{C}(X, \mathbb{R})$ for the topology of uniform convergence.

<u>Examples</u> - (1) $\mathbb{R}[X] \subset \mathcal{C}([a,b], \mathbb{R})$ satisfies the assumptions, so we recover Weierstrass's Theorem.

(2) Let $F \subset \mathcal{C}(\mathbb{S}_1, \mathbb{R})$ be defined by

$$F = \left\{ \sum_{j=-a}^{b} c_j z^j \;\middle|\; c_j \in \mathbb{Q}, \; \begin{array}{l} a \geq 0 \\ b \geq 0 \\ \text{integer} \end{array} \right\}$$

Then F is stable by sum and product, and separates points (because $z \longmapsto z$ is in F), so it is dense in $\mathcal{C}(\mathbb{S}_1, \mathbb{R})$.

In particular, since F is <u>countable</u> (because $c_j \in \mathbb{Q}$) it follows that $\mathcal{C}(\mathbb{S}_1, \mathbb{R})$ contains a countable dense subset, which is often very useful.