Chapter IV

Products, quotients, functions

1 - Product spaces

We have already seen how product sets occur naturally and they have a natural topology when the factors are top. spaces.

Def. Let I be *any* set. Let $(X_i)_{i\in I}$ be a *family* of topological spaces. The set $X = \prod_{i\in I} X_i$ is given the *product topology* defined by: $U \subset X$ is *open* if and only if for every $x = (x_i)_{i\in I}$ in U, there exists $J \subset I$ finite and $U_j \subset X_j$ open neighb. of x_j for $j \in J$ s.t.

$$\{(y_i) \mid y_j \in U_j \text{ for } j \in J\} \subset U.$$

Remarks - (1) A fundamental system of open neighb. of $x = (x_i)$ is given by these sets

$$\{(y_i) \mid y_j \in U_j \text{ for } j \in J\}$$

for $J \subset I$ finite subset, $(U_j)_{j\in J}$ neighb. of x_j.

(2) Fix $i_0 \in I$ and $U_0 \subset X_{i_0}$ open; then

$$\{(y_i) \mid y_{i_0} \in U_0\} = p_{i_0}^{-1}(U_0)$$

where

$$p_{i_0} : X \longrightarrow X_{i_0}$$

is the projection to the i_0-th component. In particular, the left-hand side being open, the projection p_{i_0} is *continuous*.

In fact:

Lemma. For <u>any</u> topological space Y, a map $f: Y \to X$ is continuous if and only if $p_i \circ f : Y \to X_i$ is continuous for all $i \in I$.

Proof. $\Rightarrow$: because $p_i \circ f$ is continuous if f is.

$\Leftarrow$: let $y \in Y$; we show continuity at y (cf. Lemma p. 17).

Let $x = (x_i) = f(y)$; pick $J \subset I$ finite and $U_j \subset X_j$ open; since $p_j \circ f : Y \to X_j$ is continuous, there is $W_j \subset Y$, ~~a~~ open neighb. of y, s.t.

$$p_j \circ f(W_j) \subset U_j$$

and then

$$\underbrace{f\Big(\bigcap_{j\in J} W_j\Big)}_{\substack{\text{open neighb. of } y \\ \text{because } J \text{ is finite}}} \subset \underbrace{\{ \underset{(z_i)}{\overset{\shortparallel}{z}} \mid z_j \in U_j, \ \forall j \in J \}}_{\text{open neighb. of } x}$$

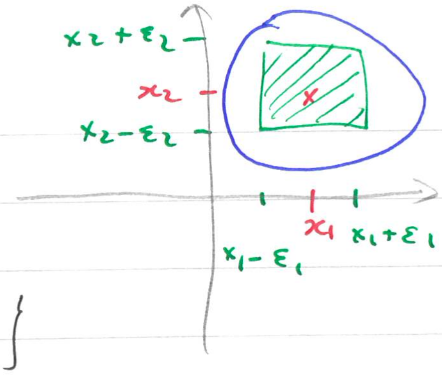
$\square$

(3) One can define another topology on $\prod X_i$ by asking for the same conditions <u>without</u> requiring J finite, which means that ~~the~~ a f.s.o.n of (x_i) is given by sets $\prod U_i$ where $U_i \subset X_i$ is an open neighb. of x_i for each i. But then the Lemma does not hold in general (if I is infinite), and this topology is much less useful.

Examples. (1) $\mathbb{R}^n$ with the euclidean topology is the same as $\mathbb{R}^n$ with the product topology of the euclidean topology of $\mathbb{R}$ on each factor. Indeed, this follows because for both, the "boxes"

$$]x_1 - \varepsilon_1, x_1 + \varepsilon_1[\times \dots \times]x_n - \varepsilon_n, x_n + \varepsilon_n[$$

form a f.s.o.n of $(x_1, \dots, x_n)$.



(2) The Cantor Space $C = \{(x_n) \mid x_n \in \{0,1\}\}$ is the infinite product space

$$\prod_{n \geq 1} X_n$$

where $X_n = \{0,1\}$ with the discrete topology for each n. This follows from the definition of the topology of C, p°14.

(3) Let X be a set and $\mathcal{F}(X)$ the set of function $X \to \mathbb{C}$ with the topology of pointwise convergence (cf. p° 15). $\mathcal{F}(X)$ can be identified with the product

$$\prod_{x \in X} X_x \qquad \text{(the } x\text{-component of } f \text{ is } f(x)\text{)}$$

where $X_x = \mathbb{C}$ for all x; then the product topology coincides with the topology of pointwise convergence.

The most important result for product spaces is:

<u>Theorem</u> (Tychonov) - If X_i is compact for all i, then $\prod X_i$ is compact with the product topology.

The proof is straightforward with ultrafilters, using the following lemma:

<u>Lemma</u> - (1) Let $\mathcal{F}$ be a filter on $\prod X_i$. Then $\mathcal{F}$ converges to $x = (x_i)$ if and only the image filter

$$p_{i*}\mathcal{F} = \{A \subset X_i \mid \exists B \in \mathcal{F},\ A \supset p_i(B)\}$$

converges to x_i for all i.

(2) Let $\mathcal{F}$ be an ultrafilter on X. For every $i \in I$, the image filter $p_{i*}\mathcal{F}$ is an ultrafilter.

Proof of the Th. (assuming the Lemma). It suffices to prove that if $\mathcal{F}$ is an ultrafilter on X, then it converges (page 39).
For each i, $p_{i*}\mathcal{F}$ is an ultrafilter on X_i (Lemma, (2)) so converges (page 39) to some x_i; then (Lemma, (1)) $\mathcal{F}$ converges to $(x_i) \in X$.
$\square$

Proof of the Lemma.
(1) $\Rightarrow$: more generally, if $f : X \longrightarrow Y$ is continuous, and $\mathcal{F}$ on X converges to x, then $f_*(\mathcal{F}) \longrightarrow f(x)$ (let U be a neighb. of $f(x)$; by continuity, there is a neighb. V of x s.t. $f(V) \subset U$; by def. of limit, there is $A \in \mathcal{F}$ s.t. $A \subset V$ and then $f(A) \in f_*(\mathcal{F})$ satisfies $f(A) \subset U$), apply this to each $p_i : X \longrightarrow X_i$.

$\Leftarrow$: let U be a neighb. of $x = (x_i)$. By definition of the product topology, we have a finite set $J \subset I$ and open neighb. $U_j \subset X_j$ of x_j s.t.
$$U \supset \{ y = (y_i) \mid y_j \in U_j \} = \bigcap_{j \in J} p_j^{-1}(U_j)$$
But $p_{j*}\mathcal{F} \longrightarrow x_j$, which implies (*) that $p_j^{-1}(U_j) \in \mathcal{F}$ for all j, and since J is finite, also
$$\bigcap_{j \in J} p_j^{-1}(U_j) \in \mathcal{F}$$
(property of filters).

[(*) : $p_{j*}\mathcal{F} \to x_j \Rightarrow U_j \in p_{j*}\mathcal{F}$
$\Rightarrow \exists A \in \mathcal{F},\ p_j(A) \subset U_j$, i.e. $A \subset p_j^{-1}(U_j)$

(2) Let $A \subset X_i$; let $B = p_i^{-1}(A) \subset X$. Since $\mathcal{F}$ is an ultrafilter, either $B \in \mathcal{F}$, and then $A = p_i(B) \in p_{i*}\mathcal{F}$ or $X - B \in \mathcal{F}$, and then $X_i - A = p_i(X - B) \in p_{i*}\mathcal{F}$.

This means that $p_i \circ \mathcal{F}$ is an ultrafilter on X_i.

$\square$

~~[illegible]~~ Ex. (1) The Cantor space is compact. Since there is a continuous surjective map $C \longrightarrow [0,1]$, it also follows that $[0,1]$ is compact.

(2) ~~Take $X_x = \{z \in \mathbb{C} / |z| \leq 1\}$ for all $x \in \mathbb{R}$. This is compact~~ In $\mathbb{R}^n$, any set C which is closed and bounded is compact: we get $M_i \geq 0$ s.t.

$$C \subset \underbrace{\prod \underbrace{[-M_i, M_i]}_{\text{compact}}}_{\text{compact}}$$

(we use here that the euclidean topology is the product topology).

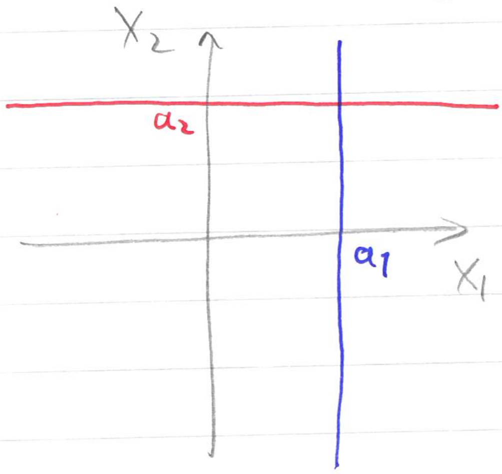
(3) Let $X_t = \{z \in \mathbb{C} \,/\, |z| \leq 1\} \subset \mathbb{C}$ for $t \in \mathbb{R}$. Then X_t is compact so

$$X = \prod_{t \in \mathbb{R}} X_t$$

is compact. This is far from easy to prove directly without ultrafilters!

(4) cf. ~~[illegible]~~ page 59

Proposition. Suppose that X_i is connected for all i. Then $\prod_{i \in I} X_i$ is connected.



Proof. Let $f : \underset{\prod X_i}{X} \longrightarrow \{0,1\}$ be continuous.

If $X = \emptyset$, there is nothing to do.

Otherwise let $a = (a_i)$ be an element of X.

For any j, the map $X_j \longrightarrow X \xrightarrow{f} \{0,1\}$ defined by

$$y \longmapsto f(\,a_i \text{ for } i \neq j,\ y \text{ at } j\text{-th place})$$

is continuous (check!), hence constant, so equal to the value $f(a)$ at $y = a_j$.

Inductively, we deduce that $f(x) = f(a)$ if x has only finitely many coordinates different from those of a.

If I is finite, this means f is constant. Otherwise, let $Y \subset X$ be the set of $x \in X$ with this property.

<u>Claim</u>: Y is dense in X, i.e., $X = \overline{Y}$.

Then we get $f^{-1}(\{f(a)\}) \supset Y$ so $f^{-1}(\{f(a)\}) \supset \overline{Y} = X$ (since it is closed), so f is constant.

<u>Proof of Claim</u>: let $x = (x_i)$ be in X and U an open neighb. of x; find $J \subset I$ finite and $U_j \subset X_j$ open neighb. of x_j s.t.

$$U \supset \{y \mid y_j \in U_j,\ \forall j \in J\} = W$$

Then note that the element (y_i) with

$$\begin{cases} y_j = x_j, & j \in J \\ y_j = a_j, & j \in I - J \end{cases}$$

belongs to $Y \cap W$, hence to $Y \cap U$, so that $Y \cap U \neq \emptyset$.

Since U is an arbitrary neighb. of x, this means $x \in \overline{Y}$, as desired.

$\square$

(4) Another example of Tychonov's Theorem: let X be a compact topological space

$\mathcal{M}(X)$ = set of probability measures (for the Borel σ-algebra) on X

There is a topology on $\mathcal{M}(X)$ s.t. a sequence (μ_n) converges to μ if

$$\forall f : X \longrightarrow \mathbb{C}, \text{ continuous, } \lim \int_X f \, d\mu_n = \int f \, d\mu .$$

The following result is fundamental:

<u>Th.</u> $\mathcal{M}(X)$ is compact.

<u>Sketch of proof</u> - The Riesz Theorem gives a bijection

$$\mathcal{M}(X) \xrightarrow{R} \mathcal{N}(X)$$
$$\mathcal{N}(X) = \left\{ \mu : \mathcal{C}(X) \to \mathbb{C} \,/\, \mu \text{ linear and } (f \geq 0 \Rightarrow \mu(f) \geq 0) \right\}$$

where a measure is mapped to the corresponding integral.

We then get an inclusion

$$\begin{cases} \mathcal{N}(X) \hookrightarrow \mathcal{F}(\mathcal{C}(X)) \\ \mu \longmapsto (\mu(f))_{f \in \mathcal{C}(X)} \end{cases}$$

which allows us to identify $\mathcal{N}(X)$ with a subspace of $\mathcal{F}(\mathcal{C}(X))$ with the topology of pointwise convergence.

Using homogeneity, we further get an injection

$$\mathcal{N}(X) \longrightarrow \prod_{\substack{f \in \mathcal{C}(X) \\ |f| \leq 1}} \overline{D}, \quad \overline{D} = \{ z \in \mathbb{C} / |z| \leq$$

which turns out to be homeomorphism on its image. By Tychonov's Th. $\prod_{|f| \leq 1} \overline{D}$ is <u>compact</u> (since $\overline{D}$ is compact), and one can check that the image of $\mathcal{N}(X)$ is <u>closed</u>, hence also compact.

$\square$