2 - Quotient topology

The last important construction of topology comes when looking at a quotient set $X/_\sim$ of a top. space X with respect to an equivalence relation $\sim$ on X.

Definition. The quotient topology on $X/_\sim$ has $U \subset X/_\sim$ open if and only if $p^{-1}(U) \subset X$ is open, where $p : X \longrightarrow X/_\sim$ is the quotient map (sending an $x \in X$ to its equivalence class).

By construction, $p : X \longrightarrow X/_\sim$ is continuous, and $C \subset X/_\sim$ is closed $\Leftrightarrow$ $p^{-1}(C)$ is closed.

Lemma. For all $\varphi : X/_\sim \longrightarrow Y$ where Y is a top. space, φ is continuous if and only if $\varphi \circ p : X \longrightarrow Y$ is continuous.

Proof. Indeed, $p^{-1}(\varphi^{-1}(U)) = (\varphi \circ p)^{-1}(U)$ for all open set $U \subset Y$, so $\varphi^{-1}(U)$ is open in $X/_\sim$ if and only if $(\varphi \circ p)^{-1}(U) \subset X$ is open

$\square$

Prop. If X is compact (resp. connected) then $X/_\sim$ is compact (resp. connected).

This is because $X/_\sim = p(X)$ and p is continuous.

On the other hand it can well happen that X is Hausdorff but not $X/_\sim$!

Examples. (1) Let $X = \mathbb{R}$ and $\sim$ the equivalence relation associated to the subgroup $\mathbb{Z} \subset \mathbb{R}$: $x \sim y \iff x - y \in \mathbb{Z}$.
We claim that $\mathbb{R}/\mathbb{Z} = \mathbb{R}/\sim$ is homeomorphic to the circle
$S_1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$.

Indeed, consider the map

$$e : \mathbb{R} \longrightarrow S_1$$
$$t \longmapsto e^{2i\pi t}$$

This is a continuous surjective map (by elementary analysis) and it is compatible with $\sim$:

$$x \sim y \implies e(x) = e(y)$$

because

$$e(x) = e^{2i\pi x} = e^{2i\pi y} \underbrace{e^{2i\pi(x-y)}}_{=1 \text{ since } x-y \in \mathbb{Z}} = e(y)$$

So there is a quotient map $\tilde{e} : \mathbb{R}/\mathbb{Z} \longrightarrow S_1$ such that $\tilde{e} \circ p = e$, where $p : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ is the projection.
By the previous Lemma, since $\tilde{e} \circ p = e$ is continuous, the map $\tilde{e}$ is continuous.

Moreover (Analysis again) $\tilde{e} : \mathbb{R}/\mathbb{Z} \longrightarrow S_1$ is in fact bijective. To check that it is then a homeomorphism, it suffices by the Prop. on p° 33 to check that $\mathbb{R}/\mathbb{Z}$ is compact and S_1 is Hausdorff. For S_1, this is because it is a metric space, and for $\mathbb{R}/\mathbb{Z}$, note that

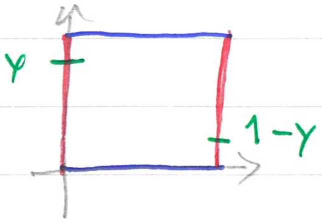
$$\mathbb{R}/\mathbb{Z} = p([0,1])$$

which is compact since $[0,1]$ is compact and p continuous.

(2) (Möbius Strip) Let $X = [0,1]^2 \subset \mathbb{R}^2$. Define

$\sim$ on X by the equivalence classes

$$\{(x,y)\} \quad \text{if} \quad 0 < x < 1$$
$$\{(0,y),(1,1-y)\} \quad \text{if} \quad 0 \leq y \leq 1$$



$$(\text{so} \quad (0,y) \sim (1,1-y))$$

The quotient space $M = X/_\sim$ is the <u>Möbius strip</u>. It is connected and compact since $[0,1]^2$ is; one can check that it is Hausdorff.

(3) Let $X = \mathbb{R}$ and define $\sim$ by
$$(\; x \sim y \iff x - y \in \mathbb{Q}).$$
Then $X/_\sim = \mathbb{R}/_\mathbb{Q}$ is an uncountable set but the quotient topology has only $\emptyset$ and $\mathbb{R}/\mathbb{Q}$ open! In particular, it is not Hausdorff at all!

Indeed, let $U \neq \emptyset$ be an open set of $\mathbb{R}/\mathbb{Q}$. Let $U' = p^{-1}(U)$, then $U' \subset \mathbb{R}$ is open (by definition) and $U' = \mathbb{Q} + U'$ (since for $x \in U'$, we get $p(x+y) = p(x) \in U$ for all $y \in \mathbb{Q}$). But then $U' = \mathbb{R}$, and so $U = p(U') = \mathbb{R}/\mathbb{Q}$.

Indeed, U' contains an open interval $I \neq \emptyset$, and for any $x \in \mathbb{R}$, we can find $y \in \mathbb{Q}$ s.t. $x - y \in I$ so $x \in y + I \subset U'$)

So a space like $\mathbb{R}/\mathbb{Q}$ cannot be studied using classical topology at all.

(4) Let X be any top. space. If $\sim$ is the relation with classes the connected components, one gets $X/_\sim = \{\text{connected components}\}$. One can check that it is completely disconnected (connected comp. of any $x \in X/_\sim$ is $\{x\}$); if X is locally connected, the connected components are open, and $X/_\sim$ has the discrete topology.