__Example__. (1) $X = S_1$: $\gamma_n(t) = e^{2i\pi n t}$, $0 \leq t \leq 1$, is a closed path in X for all $n \in \mathbb{Z}$. It turns around the circle n times if $n \geq 1$ (counterclockwise).

(2) Let X be the Cantor space. Any path $\gamma : \mathbb{I} \to C$ is __constant__ since $\gamma(\mathbb{I})$ must be connected in C.

Because of examples like (2), paths are usually only useful for spaces with lots of maps from $\mathbb{I}$ to X, in the sense for instance that any two points can be joined by a path.

__Def.__ X is __path-connected__ if for all x, x' in X, there is a $\gamma : \mathbb{I} \longrightarrow X$ continuous with $\gamma(0) = x$ and $\gamma(1) = x'$.

X is locally path-connected if for all x in X, there is a fund. system of path-connected neighbs of x.

__Ex.__ (1) $\mathbb{R}^n$ is path-connected and locally path-connected for all $n \geq 0$.

(2) Any top. manifold is locally path-connected.

(3) Any path-connected space is connected: for $f : X \to D$ continuous, where D is discrete, and for any x, x' in X, we find $\gamma : \mathbb{I} \longrightarrow X$ with $\gamma(0) = x$, $\gamma(1) = x'$, and then

$$f \circ \gamma : \mathbb{I} \longrightarrow D$$

is continuous, hence constant, so

$$f(\gamma(0)) = f(x)$$
$$\parallel$$
$$f(\gamma(1)) = f(x')$$

So f is constant.

It is __not true__ however that connected implies path-connected.
(see exercises 11)

Example. Let $h : X \times \mathbb{I} \longrightarrow Y$ be a homotopy from f_0 to f_1. We can interpret it as a <u>path</u> in the space $\mathscr{C}(X, Y)$: define $\gamma(t)$, $0 \leq t \leq 1$, to be the function such that

$$\gamma(t)(x) = h(x, t)$$

so that

$$\gamma : \mathbb{I} \longrightarrow \mathscr{C}(X, Y)$$

is a well-defined map. One can show that γ is <u>continuous</u> when $\mathscr{C}(X, Y)$ has the compact-open topology (see p° 68). Moreover, the map thus defined

$$\mathscr{C}(X \times \mathbb{I}, Y) \longrightarrow \mathscr{C}(\mathbb{I}, \mathscr{C}(X, Y))$$

is a bijection <u>if X is locally compact.</u>

3. <u>The fundamental group</u>

<u>Def.</u> Let X be a topological space. Two paths $\gamma_0, \gamma_1 : \mathbb{I} \longrightarrow X$ are (strictly homotopic / path homotopic) if there exists a homotopy

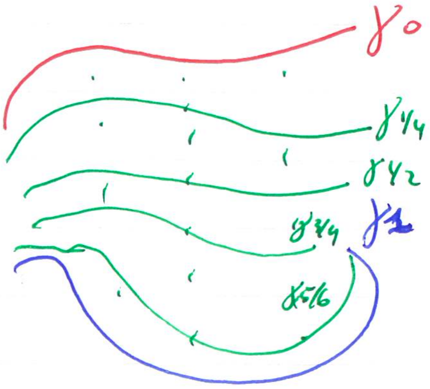
$$h : \mathbb{I} \times \mathbb{I} \longrightarrow X$$

from γ_0 to γ_1 such that the starting point of the paths

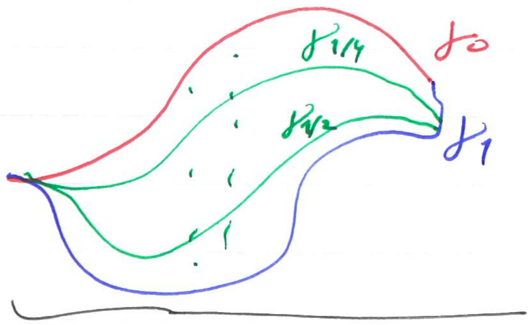
$$s \longmapsto h(s, t)$$

are constant for $0 \leq t \leq 1$, and similarly for the end point.

In particular, γ_0, γ_1 must join the same points



<u>not</u> strict



<u>strict</u>

Lemma - The relation "γ_0 is path-homotopic to γ_1" is an equivalence relation on the set of paths in X.

~~[illegible]~~ If γ_0, γ_1 are paths in X which can be composed ($\gamma_0(1)=\gamma_1(0)$), then if γ_0', γ_1' are respectively path-homotopic to γ_0, γ_1, the composed paths $\gamma_0 * \gamma_1$ and $\gamma_0' * \gamma_1'$ are path-homotopic

Proof. This is analogue to that of the Prop. on p° 77.

$\square$

Thus, if we denote by $\Lambda_{x,y}$ the set of path-homotopy classes of paths from x to y, we have well-defined maps

$$\begin{cases} \Lambda_{x,y} \times \Lambda_{y,z} \longrightarrow \Lambda_{x,z} \\ (\gamma_0, \gamma_1) \longmapsto \gamma_0 * \gamma_1 \end{cases}$$

Proposition - Denote by ε_x the constant path $t \mapsto x$ for all $x \in X$, so $\varepsilon_x \in \Lambda_{x,x}$.

(1) We have $\gamma_0(\gamma_1\gamma_2) = (\gamma_0\gamma_1)\gamma_2$ for $\gamma_0 \in \Lambda_{x,y}$, $\gamma_1 \in \Lambda_{y,z}$, $\gamma_2 \in \Lambda_{z,w}$

(2) We have $\begin{cases} \varepsilon_x \gamma = \gamma & \text{for } \gamma \in \Lambda_{x,y} \\ \gamma \varepsilon_y = \gamma & \text{for } \gamma \in \Lambda_{xy} \end{cases}$

(3) We have $\varepsilon_y = \bar{\gamma}\gamma$; $\gamma\bar{\gamma} = \varepsilon_x$ for $\gamma \in \Lambda_{xy}$ (where $\bar{\gamma}$ is the class of the reversed path : $\bar{\gamma} \in \Lambda_{yx}$)

Corollary - Fix $x \in X$. Then $\Lambda_{x,x}$ is a group with operation $(\gamma_0, \gamma_1) \longmapsto \gamma_0\gamma_1$, unit ε_x, inverse $\gamma \mapsto \bar{\gamma}$.

Def. $\Lambda_{x,x}$ is called the fundamental group of X at x, and is also denoted $\pi_1(X, x)$.

<u>Proof of the proposition.</u> (1) Spelling out the definition, if $\gamma_0, \gamma_1, \gamma_2$ are actual paths, then

$$\gamma_0(\gamma_1\gamma_2)(t) = \begin{cases} \gamma_0(2t), & 0 \le t \le 1/2 \\ \gamma_1(4t-2), & 1/2 \le t \le 3/4 \\ \gamma_2(4t-3), & 3/4 \le t \le 1 \end{cases}$$

and

$$(\gamma_0\gamma_1)\gamma_2(t) = \begin{cases} \gamma_0(4t), & 0 \le t \le 1/4 \\ \gamma_1(4t-1), & 1/4 \le t \le 1/2 \\ \gamma_2(2t-1), & 1/2 \le t \le 1 \end{cases}$$

So it is a matter of shifting these intervals continuously. For instance, take

$$h(t,s) = \begin{cases} \gamma_0\left(\dfrac{4t}{s+1}\right), & 0 \le t \le \dfrac{s+1}{4}, \\ \gamma_1(4t-s-1), & \dfrac{s+1}{4} \le t \le \dfrac{s+2}{4}, \\ \gamma_2\left(\dfrac{4t-s-2}{2-s}\right), & \dfrac{s+2}{4} \le t \le 1 \end{cases}$$

Note that $\begin{cases} h(0,s) = \gamma_0(0) = x & \text{for all } s \\ h(1,s) = \gamma_2(1) = w \end{cases}$

and check that h is continuous, so this is a path-homotopy; moreover

$$h(t,0) = \begin{cases} \gamma_0(4t), & 0 \le t \le 1/4 \\ \gamma_1(4t-1), & 1/4 \le t \le 1/2 \\ \gamma_2(2t-1), & 1/2 \le t \le 1 \end{cases}$$

$$= (\gamma_0\gamma_1)\gamma_2(t)$$

and $h(t,1) = \gamma_0(\gamma_1\gamma_2)(t)$.

(2) Here we have

$$\varepsilon_x \gamma(t) = \begin{cases} x, & 0 \le t \le 1/2 \\ \gamma(2t-1), & 1/2 \le t \le 1 \end{cases}$$

and we use the homotopy

$$h(t,s) = \begin{cases} x, & 0 \le t \le \frac{1-s}{2} \\ \gamma\left(\frac{2}{1+s}t + 1 - \frac{2}{1+s}\right), & \frac{1-s}{2} \le t \le 1 \end{cases}$$

to see that $\varepsilon_x \gamma = \gamma$ in $\Lambda_{x,y}$.
Similarly for $\gamma \varepsilon_x = \gamma$.

(3) ~~define~~ Now

$$\gamma\bar{\gamma}(t) = \begin{cases} \gamma(2t), & 0 \le t \le 1/2 \\ \gamma(2-2t), & 1/2 \le t \le 1 \end{cases}$$

and use

$$h(t,s) = \begin{cases} \gamma(2st), & 0 \le t \le 1/2 \\ \gamma(2s(1-t)), & 1/2 \le t \le 1 \end{cases}$$

to get $\begin{cases} h(t,0) = x & \text{for all } t \\ h(t,1) = \gamma\bar{\gamma}(t) \end{cases}$

so the path homotopy from ε_x to $\gamma\bar{\gamma}$.

Then note that $\bar{\gamma}\gamma = \bar{\gamma}\bar{\bar{\gamma}} = \varepsilon_x$.

$\square$

Example -

(1) Let $X = \mathbb{S}_1$. We have paths

$$\gamma_n : \mathbb{I} \longrightarrow \mathbb{S}_1$$
$$t \longmapsto e^{2i\pi n t}$$

for all $n \in \mathbb{Z}$. These are all loops from 1 to itself, so they define elements in $\pi_1(\mathbb{S}_1, 1)$.

Theorem - The map
$$\begin{cases} \mathbb{Z} \longrightarrow \pi_1(\mathbb{S}_1, 1) \\ n \longmapsto \gamma_n \end{cases}$$
(path-homotopy class)

is a group isomorphism.

This is not obvious, and we will give the full proof later. We can however at least show that $\pi_1(\mathbb{S}_1, 1)$ is not reduced to $\varepsilon_1 = \gamma_0$ by checking that γ_1 is not path homotopic to γ_0.

Indeed, suppose we have $h : \mathbb{I} \times \mathbb{I} \longrightarrow \mathbb{S}_1$ giving such a path homotopy. Using the strictness condition
$$h(0, s) = h(1, s) = 1 \quad \text{for all } s,$$
h defines a quotient map
$$\tilde{h} : \mathbb{I}/_\sim \times \mathbb{I} \longrightarrow \mathbb{S}_1$$
where $\sim$ is the equivalence relation with $0 \sim 1$ and no other identification. The map $t \mapsto e^{2i\pi t}$ gives an homeomorphism $\mathbb{I}/_\sim \longrightarrow \mathbb{S}_1$, and similarly $\mathbb{I}/_\sim \times \mathbb{I}$ is homeomorphic to $\mathbb{S}_1 \times \mathbb{I}$. So we get
$$\tilde{h} : \mathbb{S}_1 \times \mathbb{I} \longrightarrow \mathbb{S}_1$$
which is a homotopy between

$(x = e^{2i\pi\alpha})$
$$\tilde{h}(x, 0) = \tilde{h}(e^{2i\pi\alpha}, 0) = h(\alpha, 0) = \gamma_1(\alpha) = e^{2i\pi\alpha} = x$$
$$\tilde{h}(x, 1) = \tilde{h}(e^{2i\pi\alpha}, 1) = \gamma_0(\alpha) = 1.$$

This would mean that $\mathbb{S}_1$ is contractible.

(2) $\mathbb{R}^n$ has $\pi_1(\mathbb{R}^n, x) = \{\varepsilon_x\}$ for all x, as does any convex subset of $\mathbb{R}^n$.
$(\neq \emptyset)$

In fact:

<u>Prop.</u> Suppose X is contractible. Then for all x_0 s.t. Id_X is homotopic to x_0, we have $\pi_1(X, x_0) = \{\varepsilon_{x_0}\}$.

<u>Proof.</u> Let $h : X \times \mathbb{I} \longrightarrow X$ be a homotopy with $h(x,0) = x$, $h(x,1) = x_0$.

We define first

$$\tilde{h} : \mathbb{I} \times \mathbb{I} \longrightarrow X$$
$$(t,s) \longmapsto h(\gamma(t), s)$$

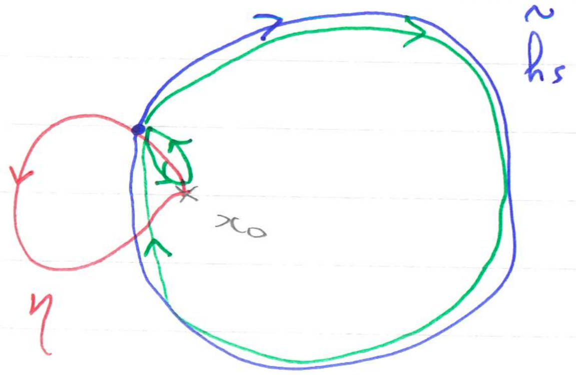
Then $\tilde{h}$ is continuous and satisfies

$$\tilde{h}(t,0) = h(\gamma(t), 0) = \gamma(t)$$
$$\tilde{h}(t,1) = h(\gamma(t), 1) = x_0$$

so it is a homotopy from γ to ε_{x_0}, but it may not be <u>strict</u>:
$$\begin{cases} \tilde{h}(0,s) = h(\gamma(0), s) = h(x_0, s) \\ \tilde{h}(1,s) = h(\gamma(1), s) = h(x_0, s) \end{cases}$$

so the endpoints are not always x_0. However, if $\eta(s) = h(x_0, s)$, we have $\eta \in \Lambda_{x_0, x_0}$ and the path $\tilde{h}_s : t \longmapsto \tilde{h}(t,s)$ is a <u>closed</u> path from $\eta(s)$ to $\eta(s)$.

We now construct another homotopy from γ to ε_{x_0} :



$$\hat{h}(t,s) = \begin{cases} \eta(3ts), & 0 \le t \le \frac{1}{3} \\ \tilde{h}(3t-1, s), & \frac{1}{3} \le t \le \frac{2}{3} \\ \eta(-3ts + 3s), & \frac{2}{3} \le t \le 1 \end{cases}$$

This is a continuous map $\mathbb{I} \times \mathbb{I} \longrightarrow X$ (since $\eta(s) = \tilde{h}(0,s) = h(x_0, s)$), and

$$\hat{h}(t,0) = \begin{cases} x_0, & 0 \le t \le 1/3 \\ \gamma(3t-1), & 1/3 \le t \le 2/3 \\ x_0, & 2/3 \le t \le 1/3 \end{cases}$$

$$\hat{h}(t,1) = \begin{cases} \eta(3t), & 0 \le t \le 1/3 \\ x_0, & \frac{1}{3} \le t \le 2/3 \\ \eta(3(1-t)), & \frac{2}{3} \le t \le 1 \end{cases}$$

with

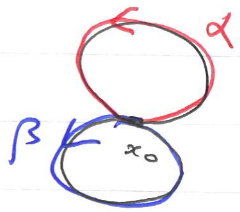
$$\begin{cases} \hat{h}(0,s) = \eta(0) = x_0 \\ \hat{h}(1,s) = \eta(0) = x_0 \end{cases}$$

One can see that $\begin{cases} \hat{h}_0 = \varepsilon_{x_0} \gamma \varepsilon_{x_0} = \gamma & \text{in } \pi_1(X, x_0) \\ \hat{h}_1 = \eta\, \varepsilon_{x_0} \bar{\eta} = \varepsilon_{x_0} & \text{in } \pi_1(X, x_0) \end{cases}$

so $\hat{h}$ really gives what we want: a path homotopy from γ to ε_{x_0}.

$\square$

(3) In general, computing $\pi_1(X, x_0)$ is not straightforward. However, for many interesting spaces, this has been done (we will see some of the underlying techniques later), and this shows that looking at fundamental groups can give <u>very interesting</u> example of groups.

For instance, for the space 

(the union of two circles in $\mathbb{C}$ which are tangent at a point), the fundamental group $\pi_1(X, x_0)$ is what is known as a <u>free group</u> on two generators: its elements are arbitrary finite products of $\alpha, \alpha^{-1}, \beta, \beta^{-1}$, manipulated according to the rule of group theory, and without any "simplification" excepts $\alpha . \alpha^{-1} = \alpha^{-1} . \alpha = e = \beta . \beta^{-1} = \beta^{-1} . \beta$
(In particular, this <u>not at all</u> an abelian group!)
There remain <u>many</u> active/open research questions about this!

One of the key property of the fundamental group is its compatibility with continuous maps.

<u>Proposition</u>. Let X, Y be top. spaces, $x_0 \in X$. Let $f : X \longrightarrow Y$ be continuous.
There exists a well-defined group morphism
$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$$
such that the path-homotopy class of $\gamma : \mathbb{I} \longrightarrow X$ is mapped to the path-homotopy class of $f \circ \gamma : \mathbb{I} \longrightarrow Y$.

<u>Proof</u>. To see that f_* is set-theoretically well-defined, we need to check that if $\gamma \sim \gamma'$ are path-homotopic, so are $f\circ\gamma \sim f\circ\gamma'$: but that is clear since if h is a

path-homotopy from γ to γ', it is immediately checked that $f\circ h : \mathbb{I}\times\mathbb{I} \longrightarrow Y$ is a path-homotopy from $f\circ\gamma$ to $f\circ\gamma'$.

Then, to see that f_* is a ~~group~~ morphism, we need to check that if γ, γ' are ~~paths~~ a loops in X at x_0, then
$$f\circ(\gamma * \gamma') \sim f\circ\gamma * f\circ\gamma'$$
But in fact, we check that
$$f\circ(\gamma*\gamma')(t) = \begin{cases} f(\gamma(2t)), & 0\le t\le 1/2 \\ f(\gamma'(2t-1)), & 1/2\le t\le 1 \end{cases}$$
$$= (f\circ\gamma) * (f\circ\gamma')\ (t).$$

$\square$

Prop. (1) We have for $\underset{x_0}{X} \xrightarrow{f} \underset{f(x_0)}{Y} \xrightarrow{g} \underset{g\circ f(x_0)}{Z}$ the equality
$$(g\circ f)_* = g_* \circ f_*.$$
(2) For any X, $x_0\in X$, $(Id_X)_* = Id_{\pi_1(X,x_0)}$.

Cor. If $f: X\to Y$ is a homeomorphism, then f_* is an isomorphism

Example. Let $X = S_1$ and $f: S_1 \longrightarrow S_1$ be the map $f(z) = z^a$ for some $a\in\mathbb{Z}$.

Then
$$f_*(\gamma_n) = \gamma_{an}$$
for all $n\in\mathbb{Z}$, so f_* "is" the morphism $n\mapsto an$ on $\mathbb{Z}$.

As an application, we have:

Prop. Let X, Y be topological space, $(x_0,y_0)\in X\times Y$. Let ~~[illegible]~~ $p: X\times Y\longrightarrow X$ and $q: X\times Y\longrightarrow Y$ be the projections. Then the morphism

$$\alpha = (p_*, q_*) : \pi_1(X \times Y, (x_0, y_0)) \longrightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

is an isomorphism.

<u>Remark.</u> Note that this is <u>more information</u> than just the <u>existence</u> of an isomorphism, since it gives a specific one.

<u>Proof.</u> 1) α is <u>injective</u>: let $\gamma : I \longrightarrow X \times Y$ be a path such that $\alpha(\gamma) = (\varepsilon_{x_0}, \varepsilon_{y_0})$. This means there are path homotopies

$$\begin{cases} h_1 : I \times I \longrightarrow X & \text{from } p\circ\gamma \text{ to } \varepsilon_{x_0} \\ h_2 : I \times I \longrightarrow Y & \text{from } q\circ\gamma \text{ to } \varepsilon_{y_0} \end{cases}$$

Then

$$h(t,s) = (h_1(t,s), h_2(t,s))$$

defines

$$h : I \times I \longrightarrow X \times Y$$

and it is a path homotopy from γ to $\varepsilon_{(x_0,y_0)}$: e.g.

$$h(t,0) = (h_1(t,0), h_2(t,0)) = (p\circ\gamma(t), q\circ\gamma(t)) = \gamma(t),$$

$$h(0,s) = (h_1(0,s), h_2(0,s)) = (x_0, y_0)$$

etc...

2) α is <u>surjective</u>: let $\gamma_1 : I \longrightarrow X$ and $\gamma_2 : I \longrightarrow Y$ be paths in X and Y; then

$$\gamma(t) = (\gamma_1(t), \gamma_2(t))$$

defines $\gamma : I \longrightarrow X \times Y$ with $p\circ\gamma = \gamma_1$, $q\circ\gamma = \gamma_2$ hence $\alpha(\gamma) = (\gamma_1, \gamma_2)$ in the fundamental groups.

$\square$

Finally, we consider the dependency of the fundamental group of the base point.

<u>Proposition.</u> Let X be a topological space. Let x_0, x_1 be elements of X. If there is a path α on X from x_0 to x_1, then the map

$$i_\alpha \quad \begin{cases} \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0) \\ \gamma \longmapsto \alpha \gamma \bar{\alpha} \end{cases}$$

is a group <u>isomorphism</u>.
In particular, if X is path connected, the fundamental groups $\pi_1(X, x_0)$ are all <u>isomorphic</u> as $x_0 \in X$ varies.

<u>Proof.</u> It is clear that i_α is well-defined (i.e. $i_\alpha(\gamma)$ only depends on the path-homotopy class of γ), cf. Lemma on p. 81. Then

$$\begin{aligned} i_\alpha(\gamma_1 \gamma_2) &= \alpha \gamma_1 \gamma_2 \bar{\alpha} \\ &= \alpha \gamma_1 \bar{\alpha} \; \alpha \gamma_2 \bar{\alpha} \\ &= i_\alpha(\gamma_1) \; i_\alpha(\gamma_2) \end{aligned}$$

using the $\left| \begin{array}{l} \text{associativity} \\ \text{identity} \end{array} \right|$ properties (from p° 81 again), so i_α is a group homomorphism.

Finally, we note that

$$i_\alpha \circ i_{\bar{\alpha}} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0)$$

is the identity:

$$\begin{aligned} i_\alpha \circ i_{\bar{\alpha}}(\gamma) &= \alpha \left(i_{\bar{\alpha}}(\gamma) \right) \bar{\alpha} \\ &= \alpha \bar{\alpha} \gamma \alpha \bar{\alpha} = \gamma \end{aligned}$$

and similarly $i_{\bar{\alpha}} \circ i_\alpha = \mathrm{Id}_{\pi_1(X, x_1)}$, so i_α is an isomorphism with inverse $i_{\bar{\alpha}}$.

□

Remark: one must be careful <u>not</u> to identify all fundamental groups without thinking! The point is that the isomorphism i_α <u>depends</u> on the choice of α, and a different choice can lead to a <u>different</u> isomorphism.

Indeed, let η be another path from x_0 to x_1. Then

$$\begin{aligned} i_\alpha(\gamma) &= \alpha \gamma \bar{\alpha} \\ &= (\alpha \bar{\eta})\, \eta \gamma \bar{\eta}\, (\eta \bar{\alpha}) \\ &= (c \circ i_\eta)(\gamma) \end{aligned}$$

where $c(\gamma) = (\alpha\bar{\eta})\,\gamma\,(\eta\bar{\alpha}) = (\alpha\bar{\eta})\,\gamma\,(\alpha\bar{\eta})^{-1}$ is the <u>conjugation</u> by the loop $\alpha\bar{\eta} \in \pi_1(X, x_0)$. If c is not the identity, then $i_\alpha \neq i_\eta$.

Example. For $n \geq 1$, let $X = \mathbb{R}^n/\mathbb{Z}^n$, with the quotient topology. Then X is homeomorphic to $(\mathbb{R}/\mathbb{Z})^n = (S_1)^n$, so $\pi_1(X, x_0)$ is isomorphic to $\mathbb{Z}^n$.