Chapter V

Homotopy and the fundamental group

1 – Introduction

We now start a very different part of the course, with a very different focus. We will introduce the study of (certain classes of) topological spaces using algebraic invariants. We can give a few motivations for this:

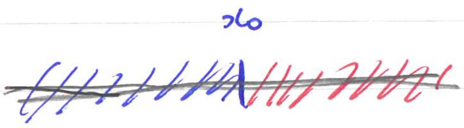
(1) a natural question from the def. of top. manifolds is: can $\mathbb{R}^n$ be homeomorphic to $\mathbb{R}^m$ if $n \neq m$? It is a fact that there are <u>continuous</u> surjective maps $\mathbb{R}^n \to \mathbb{R}^m$ if $n, m \geq 1$. (A variant: it is easy to check that

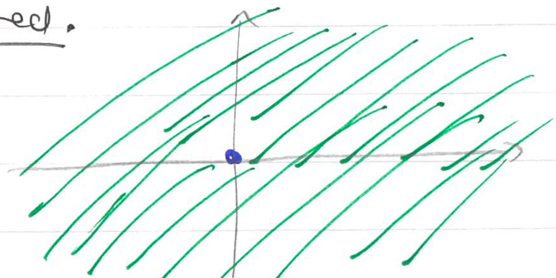
$$C \longrightarrow C \times C$$
$$(x_n) \longmapsto ((x_{2n}), (x_{2n+1}))$$

is continuous and surjective.)

The result is true (so "dimension" is well-defined), but trying to prove it with purely top. arguments is quite hard, and one idea that turned out to be very successful is to "linearize" the problem: associate to $\mathbb{R}^n$ (or other spaces) some algebraic data that is computable, and distinguishes $\mathbb{R}^n$ from $\mathbb{R}^m$ if $n \neq m$.

<u>Ex.</u> $\mathbb{R}$ is <u>not</u> homeomorphic to $\mathbb{R}^n$ for $n \neq 1$ because $\mathbb{R} - \{0\}$ (or any other $x_0 \in \mathbb{R}$) is <u>disconnected</u> whereas $\mathbb{R}^n - \{0\}$, if $n \geq 2$, is <u>connected</u>.





What about $\mathbb{R}^2$ and $\mathbb{R}^n$, $n \geq 3$? Here also we can use $\mathbb{R}^n - \{0\}$:

$$\begin{cases} \mathbb{R}^2 - \{0\} \text{ is } \underline{\text{not}} \text{ simply connected} \\ \mathbb{R}^n - \{0\} \text{ is } \text{\_\_\_\_\_\_\_\_\_\_\_\_ if } n \geq 3 \end{cases}$$

where "~~contractible~~ simply-connected" means that ~~the space~~ closed loops can be continuously "contracted" to a single point.

(2) In complex analysis, a general form of Cauchy's Theorem takes the form

$$\int_\gamma f(z)\, dz = \int_{\gamma'} f(z)\, dz$$

if γ "can be deformed" into γ' without passing through singularities of f. The study of this relation is similar to that of contractibility.

2 – Homotopy

We denote $\mathbb{I} = [0, 1]$ with the euclidean topology. This plays an essential role in homotopy theory; we recall that this is a connected and compact space – both are crucial features.

Definition – Let X, Y be topological spaces and $f_0 : X \to Y$, $f_1 : X \to Y$ two continuous maps.

An <u>homotopy</u> from f_0 to f_1 is a continuous map

$$h : X \times \mathbb{I} \longrightarrow Y$$

with product topology

such that

$$\left.\begin{aligned} h(x, 0) &= f_0(x) \\ h(x, 1) &= f_1(x) \end{aligned}\right\} \text{ for all } x \in \mathbb{I}$$

If such an h exists, one says that f_0 and f_1 are <u>homotopic</u>.

Def. A topological space X is <u>contractible</u> if the identity map $X \longrightarrow X$ is homotopic to a <u>constant</u> map $X \longrightarrow X$, $x \longmapsto x_0$ for some $x_0 \in X$.

Examples. (1) Let $n \geq 0$. Then $\mathbb{R}^n$ is <u>contractible</u>; in fact, any two maps $f_0, f_1 : X \longrightarrow \mathbb{R}^n$ are homotopic by $h : X \times \mathbb{I} \longrightarrow \mathbb{R}^n$ defined by $h(x,t) = (1-t) f_0(x) + t f_1(x)$, which is continuous (because multiplication by scalar and addition are on $\mathbb{R}^n$) and satisfies

the fact that $\mathbb{R}^n$ is an additive group is used!

$$h(x,0) = f_0(x), \quad h(x,1) = f_1(x).$$

(2) More generally, for the same reason, any <u>convex</u> subset $V \subset \mathbb{R}^n$ is contractible.

(3) Here is a <u>fundamental</u> example:

Th. $\mathbb{S}_1 = \{ z \in \mathbb{C} \;/\; |z| = 1 \}$ is <u>not</u> contractible.

Proof. We will argue by contradiction. The key fact is: the space $\mathbb{S}_1 - \{-1\}$ is <u>homeomorphic</u> to $]-\pi, \pi[$ by the map

$$\varphi : \;]-\pi, \pi[\longrightarrow \mathbb{S}_1 - \{-1\}$$
$$t \longmapsto e^{it}$$

Now suppose h is a homotopy from a map $f_0 : \mathbb{S}_1 \longrightarrow \mathbb{S}_1$ to a <u>constant</u> $f_1 : \mathbb{S}_1 \longrightarrow \mathbb{S}_1$.

Claim: There exists a continuous map $\tilde{f}_0 : \mathbb{S}_1 \longrightarrow \mathbb{R}$ such that $f_0 = e \circ \tilde{f}_0$, where

$$e : \mathbb{R} \longrightarrow \mathbb{S}_1$$
$$t \longmapsto e^{it}$$

Assuming this, we see that if S_1 were contractible, we would deduce that $f_0 : x \longmapsto x$ is of the form

$$f_0 = e \circ \tilde{f}_0$$

i.e. that there exists $\tilde{f}_0 : S_1 \longrightarrow \mathbb{R}$ s.t.

$$z = e^{i \tilde{f}_0(z)}$$

for all $z \in S_1$. We know this is not possible, as this would give a <u>global</u> logarithm / argument on $\mathbb{C}^\times$. As a more elementary argument: the identity shows that $\tilde{f}_0$ is injective, but then

$$g(z) = \frac{\overbrace{\tilde{f}_0(z) - \tilde{f}_0(-z)}^{\text{in } \mathbb{R}}}{\underbrace{|\tilde{f}_0(z) - \tilde{f}_0(-z)|}_{\neq 0 \text{ since } \tilde{f}_0 \text{ injective}}}$$

defines

$$g : S_1 \longrightarrow \{-1, 1\}$$

which is <u>continuous</u> and surjective (since $g(-z) = -g(z)$), contradicting the fact that S_1 is connected.

<u>Proof of Claim:</u> for $t \in [0,1]$, let $f_t : x \longmapsto f(x,t) : S_1 \to S_1$. By uniform continuity of $h : \underbrace{S_1 \times \mathbb{I}}_{\text{compact}} \longrightarrow S_1$, we find $k \geq 1$ s.t.

$$|s-t| \leq \frac{1}{k} \Rightarrow \forall x \in S_1, \ |f_t(x) - f_s(x)| < 2.$$

We now prove by ~~[illegible]~~ decreasing induction that if $f_{\frac{i}{k}}$ has the property of factoring by e, then so does also $f_{\frac{i-1}{k}}$. Since $f_1 = \text{constant}$ certainly does satisfy it, it follows that $f_{0/k} = f_0$ also, proving the claim.

To check the induction, note that $\left|\frac{i}{k} - \frac{i-1}{k}\right| = \frac{1}{k}$ so

$$\forall x, \quad \left| f_{\frac{i}{k}}(x) - f_{\frac{i-1}{k}}(x) \right| < 2$$

hence

$$\forall x, \quad \left| \frac{f_{i-1/k}}{f_{i/k}}(x) - 1 \right| < 2$$

which implies that $\frac{f_{(i-1)/k}}{f_{i/k}}$ is a map $\mathbb{S}_1 \longrightarrow \mathbb{S}_1 - \{-1\}$.
Now we write

$$f_{\frac{i-1}{k}} = f_{i/k} \cdot \frac{f_{(i-1)/k}}{f_{i/k}}$$

$$= f_{i/k} \cdot \varphi \circ \underbrace{\left(\varphi^{-1} \circ \frac{f_{i-1/k}}{f_{i/k}} \right)}_{\mathbb{S}_1 \longrightarrow]-\pi, \pi[\subset \mathbb{R}}$$

$$= f_{i/k} \cdot e \circ g$$

where

$$g(x) = \varphi^{-1}\left(\frac{f_{i-1/k}(x)}{f_{i/k}(x)} \right).$$

Since by assumption $f_{i/k} = e \circ g_{i/k}$ for some continuous $g_{i/k} : \mathbb{S}_1 \longrightarrow \mathbb{R}$, we get

$$f_{\frac{i-1}{k}} = e \circ g_{\frac{i-1}{k}}$$

with

$$g_{\frac{i-1}{k}} = g_{i/k} + \varphi^{-1}\left(\frac{f_{i-1/k}}{f_{i/k}} \right)$$

$\square$

(4) Homotopies are usually only interesting for relatively nice top. spaces. For instance, consider homotopies for maps from C to C, where C is the Cantor space:

$$h : C \times \mathbb{I} \longrightarrow C$$

For any $x \in C$, note that

$$h_x : \mathbb{I} \longrightarrow C$$
$$t \longmapsto h(x,t)$$

is continuous. But since $\mathbb{I}$ is connected, its image is then connected in C, which implie that it is <u>constant</u>. So $h(x,t)$ is independent of t.

Proposition – The relation « f is homotopic to g » on the set $\mathcal{C}(X, Y)$ of continuous maps from X to Y is an equivalence relation.

The set of equivalence classes is often denoted $[X, Y]$.

Proof. (1) The "identity" homotopy
$$h(x,t) = f_0(x)$$
gives a homotopy from f_0 to f_0.

(2) If $h : X \times \mathbb{I} \longrightarrow Y$ is a homotopy from f_0 to f_1, then $\bar{h}(x,t) = h(x, 1-t)$ ——— f_1 to f_0, so the relation is ~~sym~~ reflexive.

(3) Let $h_0 : X \times \mathbb{I} \longrightarrow Y$ be a homotopy from f_0 to f_1 and $h_1 : X \times \mathbb{I} \longrightarrow Y$ a homotopy from f_1 to f_2.
Define
$$h(x,t) = \begin{cases} h_0(x, 2t), & 0 \leq t \leq 1/2 \\ h_1(x, 2t-1), & 1/2 \leq t \leq 1 \end{cases}$$
Then h is continuous (elementary) and is a homotopy from f_0 to f_2, so the relation is transitive.
$\square$

$$f, f' : X \to Y \qquad g, g' : Y \to Z$$

Lemma – If $f \sim f'$, $g \sim g'$ then $g \circ f \sim g' \circ f'$.

Proof. Given
$$h : X \times \mathbb{I} \longrightarrow Y, \quad h' : Y \times \mathbb{I} \longrightarrow Z$$
homotopies giving $f \sim f'$ and $g \sim g'$ respectively, the map
$$h'' : X \times \mathbb{I} \longrightarrow Z$$
$$(x, t) \longmapsto h'(h(x,t), t)$$
gives the homotopy $g \circ f \sim g' \circ f'$: $(x, 0)$ maps to $h'(f(x), 0)$
$= g(f(x))$

and $(x,1)$ maps to $h'(f'(x),1) = g'(f'(x))$.

$\square$

This means that we have well-defined maps

$$[X,Y] \times [Y,Z] \longrightarrow [X,Z]$$

associated to composition of homotopy classes of maps.

We now focus on <u>paths</u> in a top. space.

<u>Def.</u> Let X be a topological space. A <u>path</u> in X is a continuous map $\gamma : \mathbb{I} \longrightarrow X$; one also says it is a path from $\gamma(0)$ to $\gamma(1)$. If $\gamma(0) = \gamma(1)$, this is called a <u>closed path</u> or a <u>loop</u>.

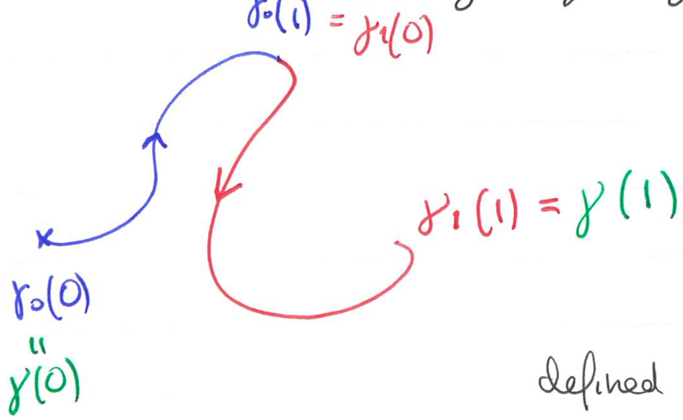
<u>Def.</u> Given paths γ_0, γ_1 in X s.t. $\gamma_0(1) = \gamma_1(0)$, the path

$$\gamma : \mathbb{I} \longrightarrow X$$
$$t \longmapsto \begin{cases} \gamma_0(2t), & 0 \leq t \leq 1/2 \\ \gamma_1(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

is well-defined from $\gamma_0(0)$ to $\gamma_1(1)$ and is called the "composition" or "juxtaposition" of γ_0 and γ_1, denoted

$$\gamma = \gamma_0 * \gamma_1, \text{ or simply } \gamma = \gamma_0\gamma_1.$$

(First go along γ_0 <u>then</u> along γ_1)



We also denote by $\bar{\gamma}$ the inverse (or opposite) of γ, defined by $\bar{\gamma}(t) = \gamma(1-t)$.

Example - (1) $X = S_1$: $\gamma_n(t) = e^{2i\pi n t}$, $0 \le t \le 1$, is a closed path in X for all $n \in \mathbb{Z}$. It turns around the circle n times if $n \geq 1$ (counterclockwise).
(2) Let X be the Cantor space. Any path $\gamma : \mathbb{I} \to C$ is constant since $\gamma(\mathbb{I})$ must be connected in C.

Because of examples like (2), paths are usually only useful for spaces with lots of maps from $\mathbb{I}$ to X, in the sense for instance that any two points can be joined by a path.

Def. X is path-connected if for all x, x' in X, there is a $\gamma : \mathbb{I} \longrightarrow X$ continuous with $\gamma(0) = x$ and $\gamma(1) = x'$.
X is locally path-connected if for all x in X, there is a fund. system of path-connected neighb of x.

Ex. (1) $\mathbb{R}^n$ is path-connected and locally path-connected for all $n \geq 0$.
(2) Any top. manifold is locally path-connected.
(3) Any path-connected space is connected: for $f : X \to D$ continuous, where D is discrete, and for any x, x' in X, we find $\gamma : \mathbb{I} \longrightarrow X$ with $\gamma(0) = x$, $\gamma(1) = x'$, and then

$$f \circ \gamma : \mathbb{I} \longrightarrow D$$

is continuous, hence constant, so

$$f(\gamma(0)) = f(x)$$
$$\shortparallel$$
$$f(\gamma(1)) = f(x')$$

So f is constant.
It is not true however that connected implies path-connected.
(see exercises 11)

Example. Let $h : X \times \mathbb{I} \longrightarrow Y$ be a homotopy from f_0 to f_1. We can interpret it as a path in the space $\mathscr{C}(X, Y)$: define $\gamma(t)$, $0 \leq t \leq 1$, to be the function such that

$$\gamma(t)(x) = h(x, t)$$

so that

$$\gamma : \mathbb{I} \longrightarrow \mathscr{C}(X, Y)$$

is a well-defined map. One can show that γ is continuous when $\mathscr{C}(X, Y)$ has the compact-open topology (see p° 68). Moreover, the map thus defined

$$\mathscr{C}(X \times \mathbb{I}, Y) \longrightarrow \mathscr{C}(\mathbb{I}, \mathscr{C}(X, Y))$$

is a bijection if X is locally compact.

3. The fundamental group

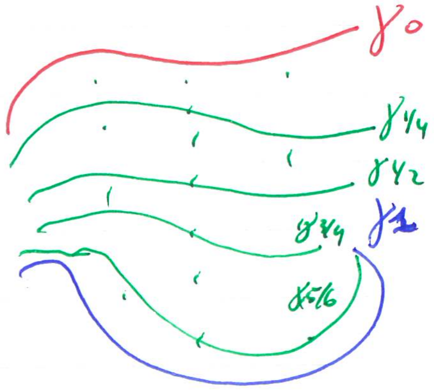
Def. Let X be a topological space. Two paths $\gamma_0, \gamma_1 : \mathbb{I} \longrightarrow X$ are (strictly homotopic / path homotopic) if there exists a homotopy

$$h : \mathbb{I} \times \mathbb{I} \longrightarrow X$$

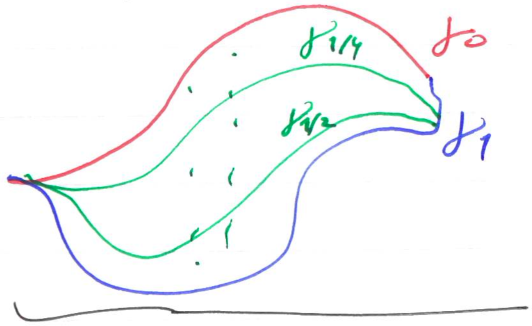
from γ_0 to γ_1 such that the starting point of the paths

$$s \longmapsto h(s, t)$$

are constant for $0 \leq t \leq 1$, and similarly for the end point. In particular, γ_0, γ_1 must join the same points.



not strict



strict

Lemma - The relation "γ_0 is path-homotopic to γ_1" is an equivalence relation on the set of paths in X.

~~[illegible]~~ If γ_0, γ_1 are paths in X which can be composed ($\gamma_0(1) = \gamma_1(0)$), then if γ_0', γ_1' are respectively path-homotopic to γ_0, γ_1, the composed paths $\gamma_0 * \gamma_1$ and $\gamma_0' * \gamma_1'$ are path-homotopic

Proof. This is analogue to that of the Prop. on p° 77.

□

Thus, if we denote by $\Lambda_{x,y}$ the set of path-homotopy classes of paths from x to y, we have well-defined maps

$$\begin{cases} \Lambda_{x,y} \times \Lambda_{y,z} \longrightarrow \Lambda_{x,z} \\ (\gamma_0, \gamma_1) \longmapsto \gamma_0 * \gamma_1 \end{cases}$$

Proposition - Denote by ε_x the constant path $t \mapsto x$ for all $x \in X$, so $\varepsilon_x \in \Lambda_{x,x}$.

(1) We have $\gamma_0(\gamma_1\gamma_2) = (\gamma_0\gamma_1)\gamma_2$ for $\gamma_0 \in \Lambda_{x,y}$, $\gamma_1 \in \Lambda_{y,z}$, $\gamma_2 \in \Lambda_{z,w}$

(2) We have $\begin{cases} \varepsilon_x \gamma = \gamma & \text{for } \gamma \in \Lambda_{x,y} \\ \gamma \varepsilon_y = \gamma & \text{for } \gamma \in \Lambda_{xy} \end{cases}$

(3) We have $\varepsilon_y = \bar{\gamma}\gamma$; $\gamma\bar{\gamma} = \varepsilon_x$ for $\gamma \in \Lambda_{xy}$ (where $\bar{\gamma}$ is the class of the reversed path : $\bar{\gamma} \in \Lambda_{yx}$)

Corollary - Fix $x \in X$. Then $\Lambda_{x,x}$ is a <u>group</u> with operation $(\gamma_0, \gamma_1) \longmapsto \gamma_0\gamma_1$, unit ε_x, inverse $\gamma \mapsto \bar{\gamma}$.

Def. $\Lambda_{x,x}$ is called the <u>fundamental group</u> of X at x, and is also denoted $\pi_1(X, x)$.

<u>Proof of the proposition.</u> (1) Spelling out the definition, if $\gamma_0, \gamma_1, \gamma_2$ are actual paths, then

$$\gamma_0(\gamma_1\gamma_2)(t) = \begin{cases} \gamma_0(2t), & 0 \le t \le 1/2 \\ \gamma_1(4t-2), & 1/2 \le t \le 3/4 \\ \gamma_2(4t-3), & 3/4 \le t \le 1 \end{cases}$$

and

$$(\gamma_0\gamma_1)\gamma_2(t) = \begin{cases} \gamma_0(4t), & 0 \le t \le 1/4 \\ \gamma_1(4t-1), & 1/4 \le t \le 1/2 \\ \gamma_2(2t-1), & 1/2 \le t \le 1 \end{cases}$$

So it is a matter of shifting these intervals continuously. For instance, take

$$h(t, s) = \begin{cases} \gamma_0\left(\frac{4t}{s+1}\right), & 0 \le t \le \frac{s+1}{4}, \\ \gamma_1(4t-s-1), & \frac{s+1}{4} \le t \le \frac{s+2}{4}, \\ \gamma_2\left(\frac{4t-s-2}{2-s}\right), & \frac{s+2}{4} \le t \le 1 \end{cases}$$

Note that $\begin{cases} h(0, s) = \gamma_0(0) = x & \text{for all } s \\ h(1, s) = \gamma_2(1) = w \end{cases}$

and check that h is continuous, so this is a path-homotopy; moreover

$$h(t, 0) = \begin{cases} \gamma_0(4t), & 0 \le t \le 1/4 \\ \gamma_1(4t-1), & 1/4 \le t \le 1/2 \\ \gamma_2(2t-1), & 1/2 \le t \le 1 \end{cases}$$

$$= (\gamma_0\gamma_1)\gamma_2(t)$$

and $h(t, 1) = \gamma_0(\gamma_1\gamma_2)(t)$.

(2) Here we have

$$\varepsilon_x \gamma(t) = \begin{cases} x, & 0 \le t \le 1/2 \\ \gamma(2t-1), & 1/2 \le t \le 1 \end{cases}$$

and we use the homotopy

$$h(t,s) = \begin{cases} x, & 0 \le t \le \frac{1-s}{2} \\ \gamma\left(\frac{2}{1+s}t + 1 - \frac{2}{1+s}\right), & \frac{1-s}{2} \le t \le 1 \end{cases}$$

to see that $\varepsilon_x \gamma = \gamma$ in $\Lambda_{x,y}$.
Similarly for $\gamma \varepsilon_x = \gamma$.

(3) ~~define~~ Now

$$\gamma\bar{\gamma}(t) = \begin{cases} \gamma(2t), & 0 \le t \le 1/2 \\ \gamma(2-2t), & 1/2 \le t \le 1 \end{cases}$$

and use

$$h(t,s) = \begin{cases} \gamma(2st), & 0 \le t \le 1/2 \\ \gamma(2s(1-t)), & 1/2 \le t \le 1 \end{cases}$$

to get $\begin{cases} h(t,0) = x & \text{for all } t \\ h(t,1) = \gamma\bar{\gamma}(t) \end{cases}$
so the path homotopy from ε_x to $\gamma\bar{\gamma}$.

Then note that $\bar{\gamma}\gamma = \bar{\gamma}\bar{\bar{\gamma}} = \varepsilon_x$.

$\square$