Chapter VI

Covering Theory

1 – Introduction

To prove that S_1 is not contractible (or, essentially equi-valently, that the loop γ_1 is non-trivial in $\pi_1(S_1, 1)$) we used the map $e: \mathbb{R} \longrightarrow S_1$, $t \longmapsto e^{2i\pi t}$: if γ_1 was trivial, we could "lift" it to $\mathbb{R}$: there would exist $\tilde{\gamma}_1$ continuous s.t.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ I \arrow[ur, dashed, "\tilde{\gamma}_1"] \arrow[r, "\gamma_1"'] & S_1 \end{tikzcd}$$

is commutative, i.e. $e \circ \tilde{\gamma}_1 = \gamma_1$. This we then showed is not the case.

The map e has quite special properties from the topologi-cal point of view, and covering theory is about generaliza-tions of this for any space X instead of S_1. As in that case, it will lead to a lot of information on the fundamental group, including a new geometric/topological perspective, and rather amazing links and analogies with field theory / Galois theory.

2 – Covering spaces

Definition – Let X be a topological space.

(1) Let D be a non-empty discrete space. The continuous map $p: X \times D \longrightarrow X$, $(x, d) \longmapsto x$ is called ~~a~~ the trivial

covering of X. ~~with fiber D, or trivial D-covering of X~~

"total space" "base space"

(2) A continuous map $f: Y \longrightarrow X$ is called a <u>covering space of X</u> if "it is locally isomorphic to a trivial covering", i.e., if for any $x \in X$, there exists an open neighborhood U of x, ~~and~~ a non-empty discrete set D and a homeomorphism

"fiber" (at x)

$$f^{-1}(U) \xrightarrow{\varphi} U \times D$$

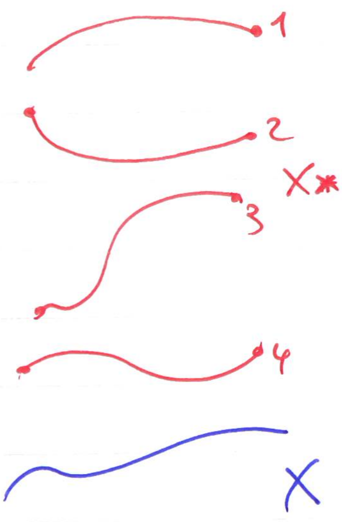
s.t.

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "f"'] & U \times D \arrow[d, "p_1"'] & (x,d) \arrow[d, mapsto] \\ U \arrow[r, "Id"] & U & x \end{tikzcd}$$

commutes.

(In particular: for $y \in U$, $f^{-1}(y)$ is in <u>bijection</u> with D, because this is so for p_1: $p_1^{-1}(y) = \{y\} \times D$, and φ induces a bijection

$$f^{-1}(y) \xrightarrow{\varphi} p_1^{-1}(y) \xrightarrow{\sim} D$$
$$z \longmapsto \varphi(z) = (y,d) \longmapsto d \quad)$$



<u>Note</u>. (1) If one can take $U = X$, then f is called a <u>trivialisable</u> covering. It means it is essentially the same as $X \times D$ for some discrete D.

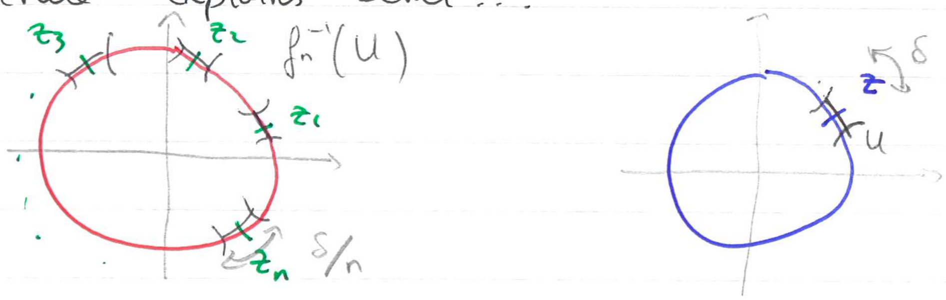
(2) If one can take the same D for all x, one speaks of a <u>D-covering</u>; if D has finite size $n \geq 1$, this is a covering <u>of degree</u> n.

Example 1 Let $X = S_1$. Then we claim that:

(1) for all $n \in \mathbb{Z}$, the map $f_n : S_1 \longrightarrow S_1$, $z \longmapsto z^n$ is a covering of degree $|n|$.

(2) the map $f : \mathbb{R} \longrightarrow S^1$, $t \longmapsto e^{2i\pi t}$ is a $\mathbb{Z}$-covering of S_1.

(1): a picture explains better...



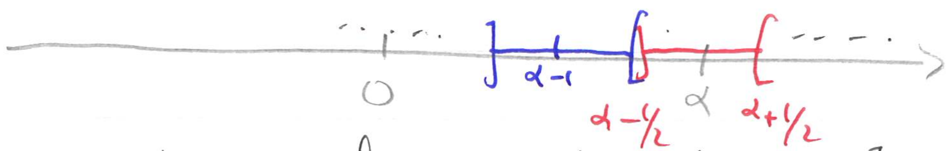
$(z_i^n = z)$ In other words: given $z \in S_1$, find $z_1, \dots, z_n$, the ~~n roots of z~~ n solutions of $z_i^n = z$; for w close enough (in argument) to z, the n roots of $w_i^n = w$ are close to $z_1, \dots, z_n$, and "do not interfere".

(2): let $z \in S_1$ and write $z = e^{2i\pi\alpha}$, as is possible by basic Analysis. Let

$$U = \left\{ e^{2i\pi t} \,\middle|\, |t - \alpha| < \frac{1}{2} \right\}$$

(for instance). Then

$$f^{-1}(U) = \bigcup_{m \in \mathbb{Z}} \left] \alpha + m - \frac{1}{2}, \alpha + m + \frac{1}{2} \right[$$



and this is homeomorphic to $]-\frac{1}{2}, \frac{1}{2}[\times \mathbb{Z}$, and then we get a homeomorphism

$$\begin{aligned} \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\times \mathbb{Z} &\longrightarrow f^{-1}(U) \\ (\beta, m) &\longmapsto \alpha + \beta + m \end{aligned}$$

and this "commutes" with the projections / f:

$$\begin{tikzcd} (\beta, m) \arrow[d, mapsto] & \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\times \mathbb{Z} \arrow[r] \arrow[d] & f^{-1}(U) \arrow[d, "f"] \\ \beta & \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\arrow[r] & U \\ & t \arrow[r, mapsto] & e^{2i\pi(\alpha+t)} \end{tikzcd}$$

(2) These examples are in fact special cases of a more general construction using <u>quotients</u> modulo group actions. Here is the statement.

<u>Theorem.</u> Let X be a topological space. Let G be a group with the discrete topology. Assume we have an <u>action</u> of G on X, i.e. a <u>continous</u> map

$$\begin{cases} G \times X \longrightarrow X \\ (g, x) \longmapsto g \cdot x \end{cases}$$

s.t. $e \cdot x = x$ for all x and $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $g_i \in G$, $x \in X$.

If the action is such that, for any $x \in X$ there exists $U \subset X$ open neighb. of x such that $gU \cap U = \emptyset$ if $g \neq e$, then the projection

$$p : X \longrightarrow X/G$$

is a covering, with fibers in bijection with G.

<u>Ex:</u> $X = \mathbb{R}$, $G = \mathbb{Z}$ with $n \cdot x = x + n$; then U can be taken to be $\left]x - \tfrac{1}{2}, x + \tfrac{1}{2}\right[$, so

$p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is a $\mathbb{Z}$-covering; composing with the homeomorphism $e : \mathbb{R}/\mathbb{Z} \to S_1$, $t \longmapsto e^{2i\pi t}$, we recover the map p of Example 7.

<u>Proof</u>. Given $\bar{x} \in X/G$, let x be an element with $p(x) = \bar{x}$, U an open neighborhood of x with the property in the statement; let $V = p(U)$. Then $p^{-1}(V) = \bigcup_{g \in G} gU$, which is a disjoint union. In particular, this is an open set (because $x \longmapsto gx$ is a homeomorphism $X \to X$, with inverse $x \longmapsto g^{-1}x$, so the image $g.U$ of U is open for all $g \in G$), so V is open in X/G and is an open neighborhood of $\bar{x}$.

Now $p^{-1}(V) = \bigcup_{g \in G} gU$ is a <u>disjoint union</u> by assumption so we have a bijection

$$\left\{\begin{array}{ccc} U \times G & \xrightarrow{\varphi} & p^{-1}(V) \\ (y, g) & \longmapsto & g \cdot y \end{array}\right.$$

and

$$\begin{tikzcd} U \times G \arrow[r, "\varphi"] \arrow[dr] & p^{-1}(V) \arrow[d, "p"] \\ & V \end{tikzcd} \qquad (y, g) \longmapsto p(y)$$

commutes. So there only remains to prove that φ is a <u>homeomorphism</u> (when G has the discrete topology). Since φ is continuous, this means we only need to check that if $W \subset U \times G$ is open, then $\varphi(W) \subset p^{-1}(V)$ is open. Since G has the discrete topology, this means checking

that if $W' \subset U$ is open and $g \in G$, then
$$\varphi(W' \times \{g\}) = gW'$$
is open, which is true (since the action is continuous as before).
$\square$

A fundamental part of covering theory will be that Example ② recovers "all" coverings, provided X has some nice suitable properties. We will see this later.

3. Lifting homotopies and applications

We will now see a link between homotopy and coverings. This comes from the "homotopy lifting property".

First some terminology: let $f : Y \longrightarrow X$ be a continuous map, and let $\varphi : Z \longrightarrow X$ be another. One says that φ lifts to Y if there is $\tilde{\varphi} : Z \longrightarrow Y$ s.t. $f \circ \tilde{\varphi} = \varphi$.

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ Z \arrow[ur, "\tilde{\varphi}"] \arrow[r, "\varphi"'] & X \end{tikzcd}$$

<u>Theorem</u>. Let $f : Y \longrightarrow X$ be a covering map. For any space Z and any homotopy $Z \times I \xrightarrow{h} X$, and for any lift $\tilde{h}_0$ of $z \longmapsto h(z, 0)$, there exists a <u>unique</u> lift $\tilde{h}$ of h which coincides with $\tilde{h}_0$ on $Z \times \{0\}$.

Before giving the proof, we use this to finish the proof that $\pi_1(S_1, \{1\})$ is isomorphic to $\mathbb{Z}$. In fact, we can do better.

Definition - A top. space X is <u>simply connected</u> if it is path connected and $\pi_1(X, x_0) = \{\varepsilon_{x_0}\}$ for all $x_0 \in X$.

Theorem - Let X be simply connected. If G is a discrete group acting continuously on X so that the projection map $p : X \longrightarrow G\backslash X$ is a G-covering space, then for all $\overline{x_0} \in G\backslash X$, the fundamental group $\pi_1(G\backslash X, \overline{x_0})$ is isomorphic to G.

So the assumption of the th. on p° 95 hold

Cor. $\pi_1(\mathbb{R}/\mathbb{Z}, 0)$ is isomorphic to $\mathbb{Z}$, and $\pi_1(S_1, 1)$ also.

<u>Proof of Th.</u> (<u>assuming homotopy lifting</u>): we construct an "explicit" morphism

$$\alpha : \pi_1(G\backslash X, \overline{x_0}) \longrightarrow G$$

as follows: for $\gamma : I \longrightarrow G\backslash X$ loop at x_0, we apply the homotopy lifting

$$h : \{\overline{x_0}\} \times I \longrightarrow G\backslash X$$
$$(\overline{x_0}, t) \longmapsto \gamma(t).$$

A lifting of h_0 is just a point $x_0 \in X$ s.t. $p(x_0) = \overline{x_0}$; in particular, it exists and can be fixed independently of γ. Now a ~~lift~~ lift $\tilde{h} : \{\overline{x_0}\} \times I \longrightarrow X$ is equivalent to a path

$$\tilde{\gamma} : I \longrightarrow X$$
$$t \longmapsto \tilde{h}(\overline{x_0}, t)$$

such that $p \circ \tilde{\gamma} = \gamma$. In particular, $p(\tilde{\gamma}(1)) = \underbrace{\gamma(1)}_{= \overline{x_0}}$

so $\tilde{\gamma}(1)$ is an element of the fiber $p^{-1}(\{\overline{x_0}\})$. This means there is a unique $g \in G$ s.t. $\tilde{\gamma}(1) = g \cdot x_0 \in X$.

$\tilde{\gamma}(1) = \beta(\gamma)x_0$

We define first $\beta : \{\text{loops at } \bar{x}_0\} \longrightarrow G$ by which is well-defined since the lift $\tilde{h}$ exists and is <u>unique</u>. (See illustration p° 100bis)

We next claim:

(i) β induces a group homomorphism $\bar{\beta} : \pi_1(\frac{X}{G}, \bar{x}_0) \longrightarrow G$

(ii) $\bar{\beta}$ is surjective

(iii) $\bar{\beta}$ is injective.

This will conclude the proof.

<u>Proof of (i)</u>: the key point is the <u>uniqueness</u> of the lift $\tilde{h}$: given γ, γ', note that

$$\tilde{\gamma}''(t) = \begin{cases} \tilde{\gamma}(2t), & 0 \le t \le \frac{1}{2} \\ \beta(\gamma)\,\tilde{\gamma}'(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

(where $\tilde{\gamma}, \tilde{\gamma}'$ lift γ and γ') is a lift of $\gamma * \gamma'$:

$$p \circ \tilde{\gamma}''(t) = \begin{cases} \gamma(2t), & 0 \le t \le \frac{1}{2} \\ \gamma'(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

and $\tilde{\gamma}''$ is continuous since $\tilde{\gamma}(1) = \beta(\gamma)x_0$. So by uniqueness, we get

$$\beta(\gamma * \gamma') = \beta(\gamma)\,\beta(\gamma').$$

So we need only check that β is well-defined on path-homotopy classes to finish proving (i): but if $\gamma \sim_p \gamma'$, then a path homotopy

$$h : \square \times \square \longrightarrow G \backslash X$$

admits a unique lifting $\tilde{h} : \square \times \square \longrightarrow X$ such that

$$\tilde{h}(0, s) = x_0$$

(since $h(0, s) = \bar{x}_0$ for all s by the path-homotopy property

Then $\eta(s) = \tilde{h}(1, s)$ defines a path on X such that

$$p(\eta(s)) = \overline{x_0}$$

for all $s \in \mathbb{I}$. This means that η is a continuous map

$$\eta : \mathbb{I} \longrightarrow p^{-1}(\{\overline{x_0}\})$$

which is constant since $\mathbb{I}$ is connected and $p^{-1}(\{\overline{x_0}\})$ ~~connected~~ discrete (prop. of covering spaces).

In particular:

$$\beta(\gamma)\, x_0 \underset{\text{def. of } \beta(\gamma)}{=} \eta(0) \underset{\eta \text{ constant}}{=} \eta(1) \underset{\text{def of } \beta(\gamma')}{=} \beta(\gamma')\, x_0$$

hence

$$\beta(\gamma) = \beta(\gamma').$$ [because the covering is a G-covering]

<u>Proof of (ii)</u>: let $g \in G$; we can find a path α on X from x_0 to $g x_0$, since X is path-connected. Then $\gamma = p \circ \alpha$ is a loop at $\overline{x_0}$ in $G \backslash X$. Its unique lift is α, so

$$\beta(\gamma)\, x_0 = \alpha(1) = g\, x_0$$

gives $\beta(\gamma) = g$. This gives the surjectivity.

<u>Proof of (iii)</u>: let γ be a loop at $\overline{x_0}$ such that $\overline{\beta}(\gamma) = e$ in G. This means that the lift $\tilde{\gamma}$ of γ satisfies

$$\tilde{\gamma}(1) = e \cdot x_0 = x_0,$$

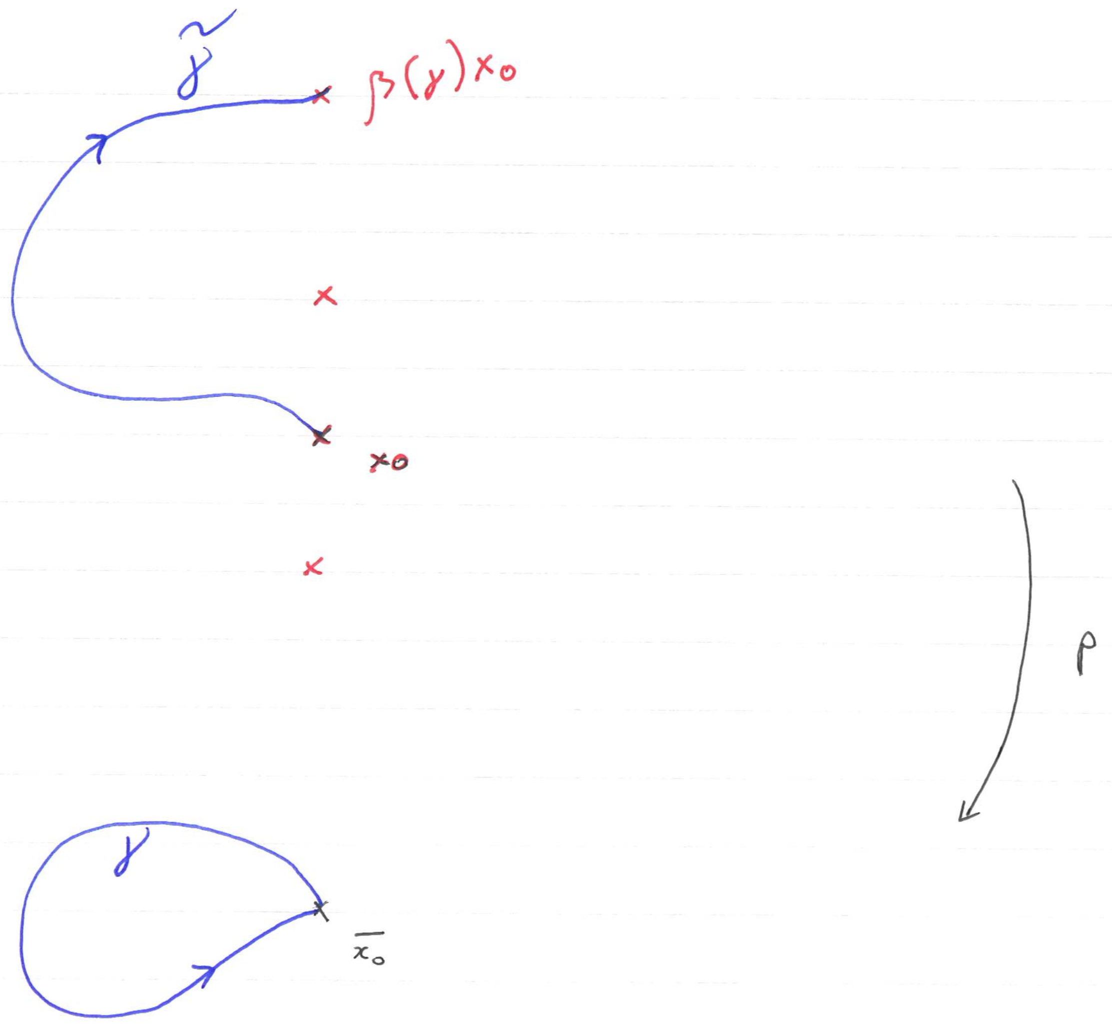
or in other words, that $\tilde{\gamma}$ is also a loop (at x_0). But since X is simply connected, there is a ~~homotopy~~ path-homotopy

$$h : \mathbb{I} \times \mathbb{I} \longrightarrow X \quad \text{from } \tilde{\gamma} \text{ to}$$

ε_{x_0}. Then $p \circ h : \mathbb{I} \times \mathbb{I} \longrightarrow G \backslash X$ is a path-homotopy from $\gamma = p \circ \tilde{\gamma}$ to $\varepsilon_{\overline{x_0}} = p \circ \varepsilon_{x_0}$. So the class of γ in $\pi_1(G \backslash X, \overline{x_0})$ is the neutral element. This means that

$$\operatorname{Ker} \overline{\beta} = \{\varepsilon_{\overline{x_0}}\}$$

so $\overline{\beta}$ is injective. $\square$


$\tilde{\gamma}$
$\beta(\gamma)x_0$
x_0
p
γ
$\bar{x}_0$

Note. In the case of $\mathbb{S}_1$, the morphism we get from the covering $\mathbb{R} \to \mathbb{S}_1 \simeq \mathbb{R}/\mathbb{Z}$ is

$$\begin{aligned} \beta : \quad \pi_1(\mathbb{S}_1, 1) &\longrightarrow \mathbb{Z} \\ \gamma &\longmapsto \tilde{\gamma}(1) \end{aligned}$$

where $\tilde{\gamma}$ lifts γ. A path that lifts $\gamma_n : t \longmapsto e^{2i\pi n t}$ is simply given by the formula $\tilde{\gamma}_n(t) = nt$ (since $e \circ \tilde{\gamma}_n = \gamma_n$), so $\beta(\gamma_n) = n$. Hence β is the inverse bijection of $n \longmapsto \gamma_n$.

Now we prove the Homotopy Lifting Property: we want to lift to Y a map $Z \times \mathbb{I} \xrightarrow{h} X$.

Step 1. If Z has a single element, then uniqueness holds.

Indeed, let $\tilde{h}$, $\tilde{h}'$ be lifts with $\tilde{h}(0) = \tilde{h}'(0)$. Then let

$$A = \{ t \in \mathbb{I} \mid \tilde{h}(t) = \tilde{h}'(t) \}.$$

Note:
(i) A is not empty ($0 \in A$)
(ii) A is closed ($\tilde{h}$, $\tilde{h}'$ are continuous, $\mathbb{I}$ Hausdorff)
(iii) A is open: let $t \in A$; let U be a neighb. of $h(t) \in X$ s.t. f is trivializable on U, and let V be a connected open neighb. of t s.t. $h(V) \subset U$. So we have a discrete space $D \neq \emptyset$ and

$$\begin{tikzcd} & f^{-1}(U) \arrow[r, "\varphi"] \arrow[d] & U \times D \arrow[dl] \\ V \arrow[r, "h"] & U & \end{tikzcd}$$

$\hat{h}$

Any lift $\hat{h}$ of h on V (e.g. $\tilde{h}$, $\tilde{h}'$ restricted to V) must satisfy $\varphi(\hat{h}(s)) = (h(s), \delta(s))$ for some $s \in V$

continuous $\delta : V \longrightarrow D$. Since V is connected, δ is constant; in particular for $\hat{h} = \tilde{h}|V$ or $\tilde{h}'|V$, δ is equal to the D-component of $\tilde{h}(t)$ or $\tilde{h}'(t)$. Since these are equal, we get $\tilde{h} = \tilde{h}'$ on V, so $V \subset A$.

Now since $\mathbb{I}$ is connected, we conclude that $A = \mathbb{I}$, and $\tilde{h} = \tilde{h}'$ everywhere.

<u>Step 2</u> - Uniqueness holds for any Z: indeed, fixing z_0 in Z, any two lifts h, $\tilde{h}'$ satisfy

$$\tilde{h}(z_0, t) = \tilde{h}'(z_0, t)$$

for all $t \in \mathbb{I}$, ~~since~~ by Step 1 with Z replaced by $\{z_0\}$.

<u>Step 3</u> - We now prove existence. First, we observe that it holds locally: given $(z_0, t_0) \in Z \times \mathbb{I}$, and an open neighb. U of z_0, ~~an open set~~ an open interval J containing t_0, if $h(U \times J)$ is contained in $W \subset X$ open s.t. f is trivializable over W, then for any lift

$$\tilde{h}_0 \text{ of } z \longmapsto h(z, t_0)$$

over U, we can find a lift

$$\tilde{h} \text{ of } h \text{ over } U \times J:$$

indeed, in the picture

$$\begin{tikzcd}
 & f^{-1}(W) \arrow[r, "\varphi"] \arrow[d, "f"'] & W \times D \arrow[dl, "p_1"] \\
U \times J \arrow[r] & W &
\end{tikzcd}$$

we can write $\varphi(\tilde{h}_0(z, t_0)) = (h(z, t_0), \delta(z))$ where

$$\delta : U \longrightarrow D$$

is continuous, and we define $\tilde{h}(z,t) = \varphi^{-1}(h(z,t), \delta(z))$.

In fact, this $\tilde{h}$ is <u>unique</u>, by the same argument as in Step 1.

Next, for given $z_0 \in Z$, we find by compactness a <u>finite</u> subset $S \subset \mathbb{I}$ and for each $s \in S$ an open interval $J_s \subset \mathbb{I}$ s.t. $\mathbb{I} = \bigcup_{s \in S} J_s$, and U_s neighb. of z_0 s.t. $h(U_s \times J_s)$ is in a trivializable neighb. of $h(z_0, s)$. Let $U = \bigcap_{s \in S} U_s$, a neighb. of z_0.

Arrange S in increasing order

$$s_0 < s_1 < \dots < s_m$$

in such a way that $J_{s_i} \cap J_{s_{i+1}} \neq \emptyset$, $0 \in J_{s_0}$, $1 \in J_{s_m}$.

Now starting from the $\tilde{h}_0$ lifting $h(z, 0)$ on $U \times \{0\}$, the previous observation gives a lift on $U \times J_{s_0}$.

Picking some $s_0' \in J_{s_0} \cap J_{s_1}$, we repeat to get existence on $U \times (J_{s_0} \cup J_{s_1})$.

Iterating gives a lift on $U \times \mathbb{I}$.

Finally, define $\tilde{h}(z, t)$ to be the value of any lift defined on $U \times \mathbb{I}$ for z_0 taken to be z in this construction. By construction / uniqueness it is a well-defined lift.

$\square$

4. The universal cover

From the previous results, it is natural to ask if the fundamental group of a (connected) top. space X can always be found by finding a covering space $Y \xrightarrow{p} X$ such that $\pi_1(X, x_0)$ acts on Y with quotient homeomorphic to X, and Y simply connected.

In general, this is not true, because the existence of Y like this implies local properties of X.

Lemma. If Y is simply connected and $p : Y \longrightarrow X$ is a covering space, then X is path-connected and moreover it is "semilocally simply-connected": for every $x \in X$, there is a neighb. U s.t. any loop in X at x which is contained in U is path-homotopic to ε_x.
[Equivalently: $\pi_1(U, x) \longrightarrow \pi_1(X, x)$ induced by $U \subset X$ is trivial.]

Proof. For x and y in X, pick $\tilde{x} \in p^{-1}(x)$, $\tilde{y} \in p^{-1}(y)$; there is a path $\tilde{\gamma}$ from $\tilde{x}$ to $\tilde{y}$ in Y; then $p \circ \tilde{\gamma}$ is a path from x to y, so X is locally connected.
because Y is path-connected

Let $x \in X$; let U be an open neighb. of x s.t. p is trivializable over U:

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\phi"] \arrow[d, "p"'] & U \times D \arrow[dl, "p_1"] \\ x \in U & \end{tikzcd}$$

Pick $d_0 \in D$; then $p : \phi^{-1}(U \times \{d_0\}) \longrightarrow U$ is a homeomorphism; any loop in U gives then gives a loop

(which exists because Y is simply connected)

in $\phi^{-1}(U \times \{d_0\}) \subset Y$; let $\tilde{h}$ be a path homotopy to $\varepsilon_{\phi^{-1}(x, d_0)}$; then $p \circ \tilde{h}$ is a path-homotopy in X from γ to ε_x. Hence $\mathcal{U}$ has the desired property.

$\square$

Def. Let X be a top. space. A covering space $p: Y \longrightarrow X$ is called a <u>universal cover</u> if Y is simply-connected.

Theorem – Let X be path-connected, locally path-connected, and semilocally simply connected.

(1) X has a universal cover $p: Y \longrightarrow X$.

(2) If $p_1: Y_1 \longrightarrow X$ and $p_2: Y_2 \longrightarrow X$ are universal covers, then for any $x \in X$ and $y_i \in Y_i$ s.t. $p_i(y_i) = x$, there is a <u>unique</u> homeomorphism

$$\phi: Y_1 \longrightarrow Y_2$$

s.t. $p_2 \circ \phi = p_1$ and $\phi(y_1) = y_2$.

(3) For any $x \in X$, $\pi_1(X, x)$ acts on Y in such a way that $\pi_1(X,x) \backslash Y$ is homeomorphic to X.

Ex. (1) $e: \mathbb{R} \longrightarrow S_1$ defined by $e(t) = e^{2i\pi t}$ is a universal cover of S_1.

(2) If X is simply connected then $Id: X \longrightarrow X$ is a universal cover.

Proof of the Theorem – (1) The idea is to view X as the space of "end points" of paths, and define Y as the space of endpoints of paths "up to homotopy".

Fix $x_0 \in X$.

So let
$$\tilde{Y} = \{ \gamma : I \longrightarrow X \mid \gamma(0) = x_0 \}$$
$$\tilde{p} : \tilde{Y} \longrightarrow X$$
$$\gamma \longmapsto \gamma(1)$$
and let
$$Y = \tilde{Y} / \sim$$
where $\sim$ is the path-homotopy relation. Since the ~~endpoint~~ endpoints are fixed in a path-homotopy, $\tilde{p}$ define a map
$$p : Y \longrightarrow X$$

Claim: with a suitable topology, p is a universal cover of X.

To prove this, it is natural to think of putting the quotient of the compact-open topology on Y, and checking first that p is continuous. This is true, but not so obvious. Instead, one can first check that p looks ~~set~~-theoretically like a covering space, and use that to ~~or~~ define the topology.

Let $V \subset X$ be any open set s.t. loops in V are path-homotopic to ε_{x_V}, where x_V is some fixed element. For $x \in V$, let $\lambda_{V,x}$ be a path in V from x_V to x. The assumption on V implie that the class $[\lambda_{V,x}]$ in $\Lambda_{x_V, x}$ is well-defined independently of the choice of $\lambda_{V,x}$.

Define $D_V = p^{-1}(\{x_V\}) \subset Y$.

Claim: the maps
$$\phi_V : p^{-1}(V) \longrightarrow V \times D_V$$
$$\psi_V : V \times D_V \longrightarrow p^{-1}(V)$$

defined by $$\begin{cases} \phi_V(\gamma) = (\gamma(1), [\gamma \overline{\lambda_{V,x}}]) \\ \psi_V(x,\eta) = [\eta\, \lambda_{V,x}] \end{cases}$$

are well-defined reciprocal bijections.
(Indeed, e.g. $\psi_V\, \phi_V(\gamma) = \psi_V(\gamma(1), \gamma\, \overline{\lambda_{V,x}})$
$= \gamma\, \overline{\lambda}_{V,x}\, \lambda_{V,x} = \gamma$.)

So we have the "structure" of a covering space! We need now to put on Y a topology so that each ϕ_V is a homeomorphism, when D_V has the discrete topology.
Because X is, by assumption, covered by open sets V as above, there is at most one topology with this property, and one can check that it exists if and only if for any two open sets V_1, V_2 (with the property that loops in V_i are homotopic to constants in X), the map
$$\phi_{V_1} \circ \psi_{V_2} : (V_1 \cap V_2) \times D_{V_2} \longrightarrow (V_1 \cap V_2) \times D_{V_1}$$
is a homeomorphism (knowing that it is a bijection, each side being in bijection with $p^{-1}(V_1 \cap V_2)$). This map is
$$(x, \eta) \longmapsto (\eta\lambda_{V_2,x}(1), \eta\, \lambda_{V_2,x} \overline{\lambda_{V_1,x}})$$
$$= (x, \eta\, \lambda_{V_2,x} \overline{\lambda_{V_1,x}}),$$
and is continuous if and only if $\eta\, \lambda_{V_2,x} \overline{\lambda_{V_1,x}}$ is locally constant in D_{V_1}. This is left as an exercise.

Now, knowing that $p: Y \longrightarrow X$ is a covering space, we check : (i) that Y is simply connected
(ii) that $\pi_1(X, x_0)$ acts on Y with quotient X.

<u>Proof of (i)</u>: first given $\gamma \in \tilde{Y}$ with class $[\gamma] \in Y$, let $\tilde{\eta} : I \longrightarrow \tilde{Y}$ be defined by

$$\tilde{\eta}(s) = (t \longmapsto \gamma(st))$$

Then the class of $\tilde{\eta}(s)$ is an element of Y and one can check that $\eta : s \longmapsto (\text{class of } \tilde{\eta}(s))$ is a path in Y from $\tilde{\eta}(0) = [\varepsilon_{x_0}]$ to $\tilde{\eta}(1) = [\gamma]$. So Y is path-connected.

To prove it is simply-connected we use:

<u>Lemma</u> Let $Y \longrightarrow X$ be <u>any</u> covering space. Let $y_0 \in Y$ and $x_0 = p(y_0) \in X$. Then the morphism

$$\pi_1(Y, y_0) \xrightarrow{p_*} \pi_1(X, x_0)$$

is injective.

<u>Proof</u>. Let $\tilde{\gamma}$ be a loop at y_0 s.t. $p \circ \tilde{\gamma} \sim \varepsilon_{x_0}$ (i.e., $[\tilde{\gamma}]$ is in the kernel of p_*). Pick a path-homotopy h from $p \circ \tilde{\gamma}$ to ε_{x_0}; there is a unique homotopy $\tilde{h}$ on Y lifting h with $\tilde{h}_0 = \tilde{\gamma}$.

Now note:

(1) since $p \circ \tilde{h}_1 = h_1 = \varepsilon_{x_0}$ is constant, and ε_{y_0} also has $p \circ \varepsilon_{y_0} = \varepsilon_{x_0}$, the uniqueness of the lift <u>of ε_{x_0}</u> to Y starting from y_0 shows that $\tilde{h}_1 = \varepsilon_{y_0}$.

(2) since $p(\tilde{h}(t,0)) = h(t,0) = x_0$ for all t, similarly, we have $\tilde{h}(t,0) = y_0$; also $\tilde{h}(t,1) = y_0$.

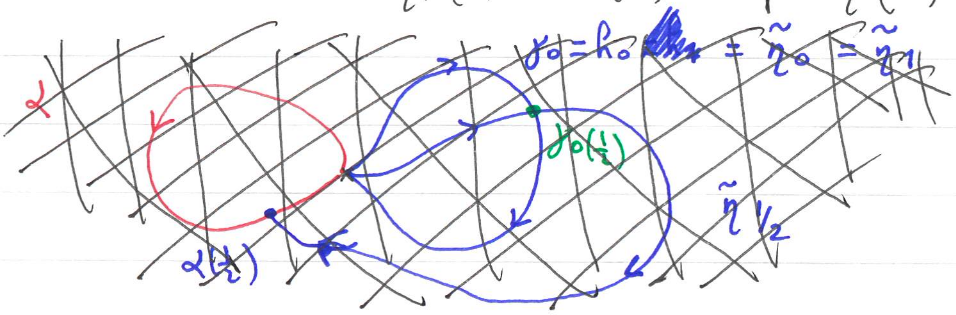
So $\tilde{h}$ is a path-homotopy from $\tilde{\gamma}$ to ε_{y_0}, hence $\tilde{\gamma}$ is trivial in $\pi_1(Y, y_0)$.

$\square$

Now let $\tilde{\eta} : I \longrightarrow Y$ be a loop at some $y_0 \in p^{-1}(x_0)$.

By the lemma, $\tilde{\eta}$ will be trivial in $\pi_1(Y, y_0)$ if $p_*(\tilde{\eta})$ is trivial in $\pi_1(X, x_0)$. $= p\circ\tilde{\eta}$

Now γ_0 is a loop at x_0, and $\tilde{\eta}_t$ is a path in X from x_0 to $\tilde{\eta}_t(1) = \alpha(t) = p\circ\tilde{\eta}(t)$.



Define $h(t,s) = \tilde{\eta}_t(s)$ for $(t,s) \in I \times I$. This is a homotopy from $t \longmapsto h(t,0) = \tilde{\eta}_t(0) = x_0$

to $t \longmapsto h(t,1) = \tilde{\eta}_t(1) = \alpha(t)$

i.e. from ε_{x_0} to α.

It has endpoints $\begin{cases} h(0,s) = \gamma_0(s) \\ h(1,s) = \gamma_0(s) \end{cases}$, so it is not a path-homotopy. However, proceeding exactly as on p°85-86 we obtain a path homotopy from ε_{x_0} to α.

Hence $p_*(\tilde{\eta}) = \varepsilon_{x_0}$, so by injectivity, $\tilde{\eta} = \varepsilon_{\gamma_0}$.

<u>Proof of (ii):</u> The action of $\pi_1(X, x_0)$ on Y is clear: for a loop η at x_0 and a path γ starting from x_0, we have also $\eta\gamma \in \tilde{Y}$, and this passes to the quotient, giving

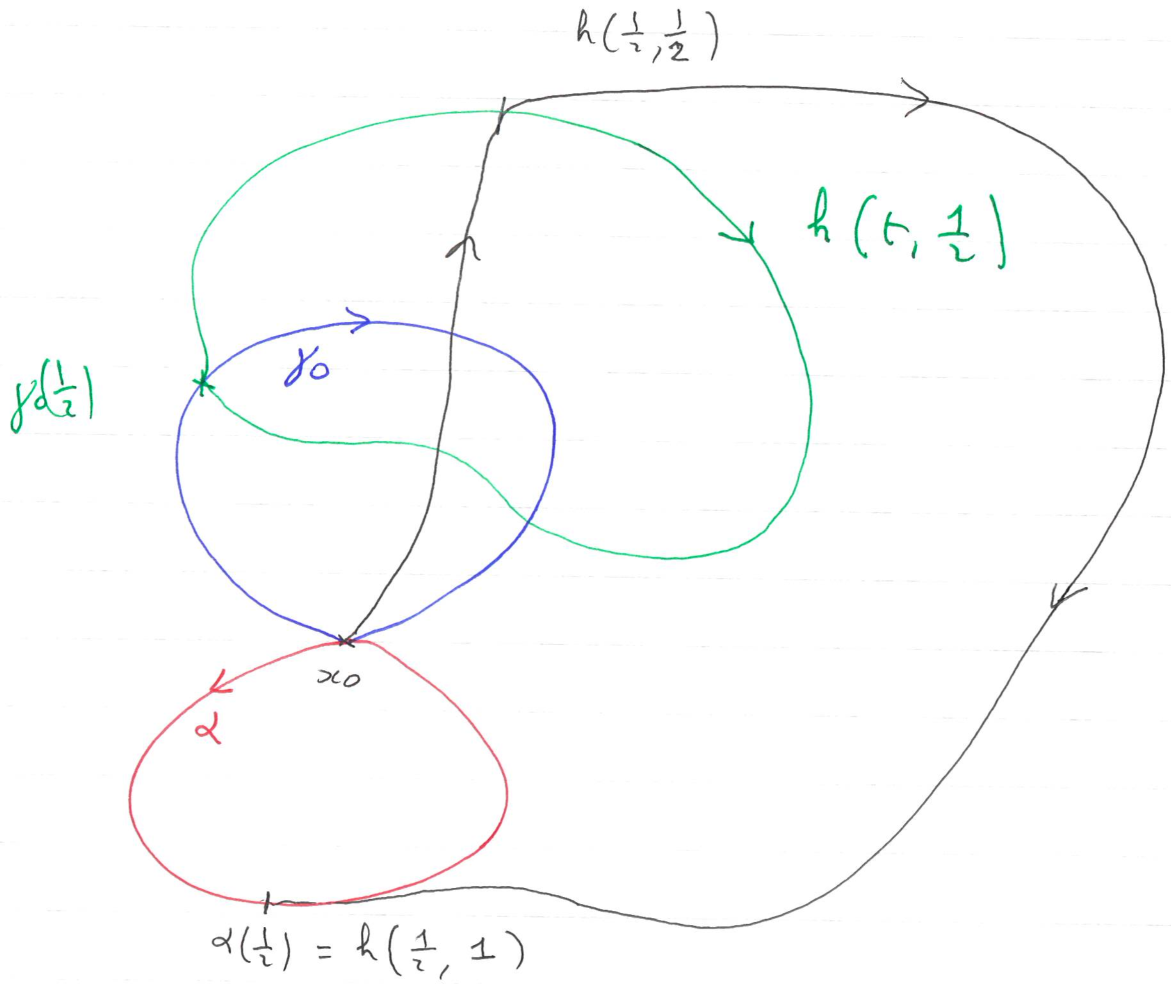
$$\pi_1(X, x_0) \times Y \longrightarrow Y.$$

Over an open set V as on p° 106 with the homeomorphism

$$\Phi_V : p^{-1}(V) \longrightarrow V \times D_V$$

The action corresponds to

$$\eta \cdot (x, \gamma) = (x, \underbrace{\eta\gamma}_{\substack{\text{path from } x_0 \text{ to } x_V \\ \text{i.e. elt of } p^{-1}(x_V)}})$$



$$\gamma_0(s) = h(0,s) = h(1,s) = \tilde{\eta}_0(s) = \tilde{\eta}_1(s)$$

$$\alpha(t) = \tilde{\eta}_t(1) = h(t,1)$$

$\tilde{\eta}_{\frac{1}{2}}$: path from x_0 to $\alpha(\frac{1}{2})$

$$h(0,\tfrac{1}{2}) = \gamma_0(\tfrac{1}{2}) = \tilde{\eta}_0(\tfrac{1}{2})$$
$$h(1,\tfrac{1}{2}) = \gamma_0(\tfrac{1}{2}) = \tilde{\eta}_1(\tfrac{1}{2})$$

$\tilde{\eta}_{\frac{1}{2}}(s) = h(\frac{1}{2}, s)$: path from x_0 to $\alpha(\frac{1}{2})$ passing through $h(\frac{1}{2},\frac{1}{2})$
=

for fixed $\gamma_0 \in p^{-1}(v)$

and since

$$\pi_1(X, x_0) \longrightarrow p^{-1}(x_v)$$
$$\eta \longmapsto \eta\gamma_0$$

is bijective with inverse

$$\gamma \longmapsto \gamma\overline{\gamma_0}$$

we deduce that the quotient is indeed X (the action is transitive on each $p^{-1}(x)$).

We now proved both (1) and (3) in the Theorem. There remains the "uniqueness" part (2), which we will discuss later...

$\square$

5 _ Classification of coverings

Why is the "universal cover" (when it exists) called universal? The reason comes from the fact that one can use it to construct all (connected) covering spaces of X. First we must explain what is meant:

Def. Let X be a top. space, and $f_1 : Y_1 \to X$, $f_2 : Y_2 \to X$ covering spaces.

A morphism of coverings from Y_1 to Y_2 is a continuous map $\varphi : Y_1 \to Y_2$ s.t.

$$f_2 \circ \varphi = f_1 .$$

$$\begin{tikzcd} Y_1 \arrow[rr, "\varphi"] \arrow[dr, "f_1"'] & & Y_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

If φ is a homeomorphism, then $\varphi^{-1} : Y_2 \to Y_1$ is a morphism of coverings, and φ is said to be an isomorphism.

Ex. $f : Y \to X$ is trivializable $\iff$ there is $D \neq \emptyset$ discrete s.t. f is isomorphic to $X \times D \xrightarrow[p_1]{} X$.

Theorem. Let X be path-connected, locally path-connected and semilocally simply-connected. Let $x_0 \in X$. Let $f : Y \to X$ be a universal covering space of X.

(1) For any subgroup $G \subset \pi_1(X, x_0)$, the orbit space $G \backslash Y$ is a covering space of X:

$$\begin{tikzcd} Y \arrow[r] \arrow[rr, bend right, "f"'] & G\backslash Y \arrow[r] & X \end{tikzcd}$$

This is a $\pi_1(X, x_0)/G$ - covering which is connected.

(2) Conversely, let $Z \xrightarrow{g} X$ be a path connected covering space. There exists a subgroup $G \subset \pi_1(X, x_0)$ such that g is isomorphic, as a covering of X, to $G \backslash Y \longrightarrow X$. In fact, $G = g_* \pi_1(Z, z_0)$ works, where z_0 satisfies $g(z_0) = x_0$.

Example. For $X = S_1$, with universal cover $\mathbb{R}$ and fundamental group $\pi_1(X, x_0) \simeq \mathbb{Z}$, the subgroups are

$$\{0\} \subset \mathbb{Z}$$
$$n\mathbb{Z} \subset \mathbb{Z}, \quad n \geq 1$$

The corresponding coverings of S_1 are

$$\begin{cases} S_1 \xrightarrow{f_n} S_1, & n \geq 1 \\ t \longmapsto t^n \end{cases}$$

and

$$\begin{cases} \mathbb{R} \longrightarrow S_1 \\ t \longmapsto e^{2i\pi t} \end{cases}, \text{ i.e. the universal cover.}$$

<u>Remark</u> - Recall ($p^\circ 108$) that since $g : Z \longrightarrow X$ is a covering space, the morphism $g_* : \pi_1(Z, z_0) \to \pi_1(X, x_0)$ is injective, so the subgroup G "is" a subgroup of $\pi_1(X, x_0)$.

<u>Corollary</u> - If X is locally path-connected and simply-connected. Then every covering $g : Z \longrightarrow X$ is trivializable.

<u>Proof</u>. Every path-connected component Y' of Y defines a path-connected covering space $f : Y' \longrightarrow X$. This is isomorphic to $G\backslash X$ for some subgroup of $\pi_1(X, x_0) = \{\varepsilon_{x_0}\}$, so homeomorphic to X, and this is only possible if f is itself an homeomorphism (covering of degree 1). So Y is homeomorphic to $X \times \{$set of path-connected components of $Y\}$ and is trivializable.

$\square$

<u>Remark</u>. The correspondance

$$\left\{ \begin{array}{c} \text{path-connected covering} \\ \text{spaces of } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{subgroups of} \\ \pi_1(X, x_0) \end{array} \right\}$$

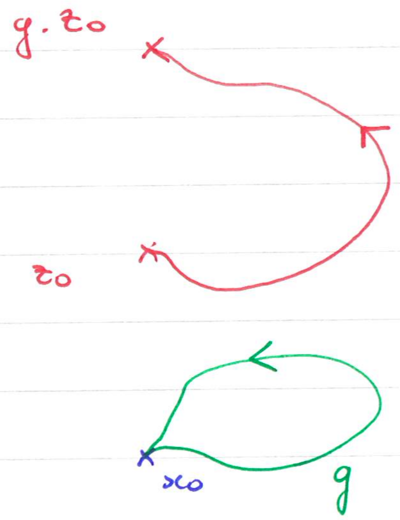
is reminiscent of Galois theory in algebra. Indeed, the analogy is even stronger if one observes that given

$$g : Z \longrightarrow X$$

the subgroup G such that Z can be identified to $G\backslash Y$ can also be identified as the <u>automorphism group</u> $\mathrm{Aut}_X(Z)$ of g as a covering space, i.e. the group of isomorphisms $\varphi: Z \to Z$ as covering spaces of X. (This identification goes by sending $g \in G$ to the automorphism $z \longmapsto g.z$ of $Z = G \backslash Y$.)

Other aspects of Galois theory have then analogues:

(i) $\pi_1(X, x_0)$ acts on the fiber $f^{-1}(\{x_0\})$ by sending (g, z_0) to the endpoint of the unique path in Z lifting a loop representing g starting at z_0.



(ii) The action is <u>transitive</u> $\Leftrightarrow$ Z is path-connected [analogue of the action of the Galois group of the splitting field of a polynomial on its roots in some algebraic closure].

(iii) There is a notion of <u>Galois covering</u> ($\Leftrightarrow$ $\mathrm{Aut}_X(Z)$ is "the same" as the fibers) with analogue behavior to Galois extensions; the universal cover is always a Galois covering, and for $G \subset \pi_1(X, x_0)$, $G \backslash Y$ is a Galois covering of X $\Leftrightarrow$ G is a <u>normal</u> subgroup of $\pi_1(X, x_0)$.

These analogies are particularly well-understood ~~when~~ in the context of <u>algebraic geometry</u>; indeed, suitably formalized both classical Galois theory and that of coverings (of alg. varieties) become special cases of a more general theory, due to Grothendieck and others in the early 1960's.