Chapter VII

Final remarks / perspectives

1. Some missing topics

Among the topics we did not have time to cover but which are often important, there are:

(1) "finiteness" conditions on a top. space: a space X is called

1a) Separable if there is a countable dense subset;

1b) First-countable if every point has a countable f.s.o.n;

1c) Second-countable if X has a countable basis of open sets.

Ex. For instance, $\mathbb{R}^n$, for $n \geq 0$, satisfies all these properties, every metric space is first-countable, and $\mathcal{F}(\mathbb{R};\mathbb{R})$, with the topology of pointwise convergence, doesn't satisfy any.

(2) topological groups and continuous actions (in greater generality than in the discussion of covering spaces)

(3) the Van Kampen Theorem to compute $\pi_1(X, x_0)$: this provides a way to compute the fundamental group of a space from those of open subsets in an open covering (often by just two sets: $X = U_1 \cup U_2$) and information on $\pi_1(U \cap V, x_{UV})$ for U, V in the covering.

This can be quite efficient, and implies for instance easily that if X is a Hausdorff compact topological manifold, then $\pi_1(X, x_0)$ is finitely-generated.

2. Some uses of topology...

We will sketch here the proofs of three theorems which do not seem to be about topology, and yet can be deduced quite nicely from topological ideas.

(I) The fundamental theorem of algebra

Th. Let $f \in \mathbb{C}[X]$ have degree $n \geq 1$. There exists $z \in \mathbb{C}$ such that $f(z) = 0$.

We may assume that f is monic.

Proof. If f doesn't have a root, then define

$$h(s,t) = \frac{f(te^{2i\pi s})/f(t)}{|f(te^{2i\pi s})/f(t)|}$$

which is a continuous map $[0,1] \times \mathbb{R} \longrightarrow \mathbb{S}_1$.

For $t = 0$, we have $h(s,0) = 1$, and for any $t \in \mathbb{R}$, $s \mapsto h(s,t)$ is a loop based at 1; also $h(0,t) = h(1,t) = 1$.

Rescaling (i.e. considering $h(s, Rt)$ for $t \in [0,1]$) we see that each $s \xmapsto{\gamma_R} h(s,R)$ is path-homotopic to $s \mapsto h(s,0)$, i.e. to ε_1.

So each γ_R is a trivial loop in $\pi_1(\mathbb{S}_1, 1)$.

However, take R very large and define

$$\hat{h}(s,t) = \frac{f_t(Re^{2i\pi s}) / f_t(R)}{|f_t(Re^{2i\pi s}) / f_t(R)|}$$

where

$$f_t(X) = X^n + t\,(f(X) - X^n), \text{ so}$$

that $f_0(X) = X^n$, $f_1(X) = f(X)$.

For R large, $\hat{h}$ is well-defined and is a path-homotopy from γ_R to $t \longmapsto e^{2i\pi n s}$. Hence also γ_R is equal to the class of $t \longmapsto e^{2i\pi n s}$ in $\pi_1(S_1, 1)$ and this we know is non-trivial when $n \neq 0$.

$\square$

(II) Picard's "Small" Theorem

Th. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be holomorphic everywhere. If there exist a, b in $\mathbb{C}$, $a \neq b$, s.t. $f(z) \notin \{a, b\}$ for all z, then f is constant.

Proof. We can view f as a holomorphic map

$$f : \mathbb{C} \longrightarrow \mathbb{C} - \{a, b\}.$$

The space $\mathbb{C} - \{a,b\}$ is certainly path-connected, locally-path-connected, semilocally simply connected, so it has a universal covering space $\omega : Y \longrightarrow \mathbb{C} - \{a, b\}$.

One can show that in fact Y is homeomorphic to

$$D = \{z \in \mathbb{C} \;/\; |z| < 1\}$$

in such a way that ω is holomorphic from D to $\mathbb{C} - \{a,b\}$.

(This is far from obvious! It is part of the theory of Riemann surfaces, but was "obvious" to Picard because ω can be described explicitly, and this had been done before him in the theory of "elliptic functions".)

By the theory of covering spaces one gets a lift of f:

$$\begin{tikzcd} & D \arrow[d, "\omega"] \\ \mathbb{C} \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & \mathbb{C}-\{a,b\} \end{tikzcd}$$

because $\mathbb{C}$ is simply-connected.

(General criterion: if $f: Y \longrightarrow X$ is a covering space, $Z \xrightarrow{g} X$ is continuous, then provided Z is path-connected and locally path-connected, a lift of g to Y exists $\Leftrightarrow$ $g_* \pi_1(Z, z_0) \subset f_* \pi_1(Y, y_0)$, where $f(y_0) = g(z_0)$ is a ~~base~~ point in X.)

Again one can check that $\tilde{f}$ is holomorphic. Then $\tilde{f}$ is a bounded ~~[illegible]~~ entire function; by Liouville's Theorem, it follows that $\tilde{f}$ is constant, so $f = \omega \circ \tilde{f}$ is also constant.

$\square$

(III) Inscribed rectangles

Recall a theorem stated on p° 2:

Theorem. Let $\gamma: S_1 \longrightarrow \mathbb{R}^2$ be an injective continuous map. There exist four points in $\gamma(S_1)$ which are the vertices of ~~[illegible]~~ a rectangle.

We will sketch a proof (H. Vaughan, 1977) which relies ultimately on topological properties of the Möbius strip.

(In fact, there will be many: R.E Schwartz proved that
(2020)

all points of $\gamma(\mathbb{S}_1)$, with ≤ 4 exceptions, are vertices of a rectangle with all other vertices also in $\gamma(\mathbb{S}_1)$.)

Sketch of proof: Let $M = \{ \{a,b\} \subset \mathbb{S}_1 \mid a \neq b \}$ be the set of *distinct* pairs (unordered) of points of $\mathbb{S}_1$. This can be given a natural topology, by interpreting M as

$$M \text{ "="} \quad (\mathbb{S}_1 \times \mathbb{S}_1 - \Delta) / \sim$$

where

$$\Delta = \{ (x,x) \in \mathbb{S}_1 \times \mathbb{S}_1 \mid x \in \mathbb{S}_1 \}$$

and

$$(a,b) \sim (c,d) \iff \begin{array}{l} (a,b) = (c,d) \\ \text{or} \\ (a,b) = (d,c) \end{array}$$

$$\iff \{a,b\} = \{c,d\}$$

so we put on M the quotient of the subspace of the product topology. One can check that M is homeomorphic to an "open" Möbius strip:

$$M \xrightarrow{\simeq} \left( ]0,1[\times [0,1] \right) / \sim$$

where

$$(t,0) \sim (1-t,1)$$

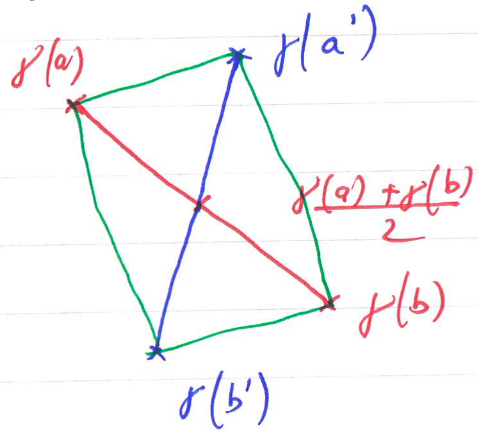
(and no other equivalences)

Now consider the map

$$\begin{aligned} \phi : \quad M &\longrightarrow \mathbb{R}^3 \\ \{a,b\} &\longmapsto \left(\frac{\gamma(a)+\gamma(b)}{2}, |\gamma(a)-\gamma(b)| \right) \end{aligned}$$

It is well-defined and continuous.

Moreover, we have a key fact: if ϕ is *not* injective, say $\phi(\{a,b\}) = \phi(\{a',b'\})$, $\{a,b\} \neq \{a',b'\}$, then $\{\gamma(a), \gamma(b), \gamma(a'), \gamma(b')\}$ are vertices of a rectangle, all on $\gamma(\mathbb{S}_1)$: the segments $[\gamma(a), \gamma(b)]$ and $[\gamma(a'), \gamma(b')]$ have the same middle point and the same length



So we need only show that ϕ is <u>not</u> injective.
To do this, observe that $\phi(M) \subset \{(x,y,z) \mid z>0\} \subset \mathbb{R}^3$.

Define

$$K = \phi(M) \cup i(\phi(M)) \cup \{(\gamma(a), 0) \mid a \in \mathbb{S}_1\} \subset \mathbb{R}^3$$

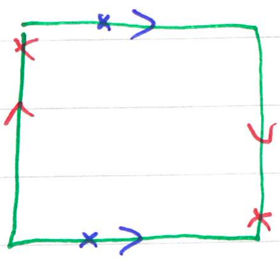
where $i(x,y,z) = (x,y,-z)$. If ϕ is injective, then ~~Then~~ one sees that K is homeomorphic, <u>in $\mathbb{R}^3$</u>, to a <u>Klein bottle</u>: union of two Möbius bands along ~~their~~ a common boundary, or equivalently

$$[0,1] \times [0,1] / \sim$$

where

$$(t,0) \sim (1-t, 1)$$
$$(0,s) \sim (1,s)$$

and no other equivalence.



<u>However</u>, a simple consequence of "standard" results in algebraic topology (Alexander duality) shows that there is no subset of $\mathbb{R}^3$ homeomorphic to K...
So ϕ could <u>not</u> be injective. $\square$