Wahrscheinlichkeit und Statistik

Serie 1

Version 2 (11. März 2024: Tippfehler in der Lösung von Aufgabe 1.1(1)(b): "$-3 \le x \le 1$" wurde durch "$-3 \le x < 1$" ausgebessert. Kleiner Tippfehler in Aufgabe 1.1(2) ausgebessert. Zusätzliche Erklärungen in der Lösung von Aufgabe 1.2(1). Die Notation wurde an die Vorlesung angepasst, wo Wahrscheinlichkeitsmasse typischerweise mit "$\mathrm{P}(\cdot)$" anstatt mit "$\mathbb{P}[\cdot]$" bezeichnet werden. Minimale Verbesserungen der Formulierungen und Formatierung.), Version 1 (1. März)

Bitte stellt Fragen in den Übungen und/oder im Forum.
Bitte stell sicher, dass du die Webseite `https://kahoot.it/` in der Übung am **05. März** öffnen kannst.
Freiwillige Abgabe bis **07. März 8:00**. Nachher kann selbstständig mit der Lösung verglichen werden.

Aufgabe 1.1 **($\liminf$ and $\limsup$ of sets)**
Let (A_n) be a sequence of events on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Recall (from measure theory) the definitions

$$\liminf_{n\to\infty} A_n := \bigcup_{n\ge 1} \bigcap_{m\ge n} A_m,$$
$$\limsup_{n\to\infty} A_n := \bigcap_{n\ge 1} \bigcup_{m\ge n} A_m.$$

Intuition: $\liminf_{n\to\infty} A_n$ consists of the elements $\omega \in \Omega$ that appear in *almost all*[1] sets A_n.
Intuition: $\liminf_{n\to\infty} A_n$ consists of the elements $\omega \in \Omega$ that appear in infinitely many sets A_n.

(1) If $\Omega = \mathbb{R}$, give the set $\limsup_{n\to\infty} A_n$ in the following three cases (please justify your answers):
  (a) $A_n = [-1/n, 3 + 1/n]$
  (b) $A_n = [-2 - (-1)^n, 2 + (-1)^{n+1})$
  (c) $A_n = p_n\mathbb{N}$, where $(p_n)_{n\ge 1}$ is the sequence of prime numbers and $p_n\mathbb{N}$ denotes the set of all multiples of p_n.

(2) Show that

$$\mathrm{P}\left(\liminf_{n\to\infty} A_n\right) \le \liminf_{n\to\infty} \mathrm{P}(A_n) \le \limsup_{n\to\infty} \mathrm{P}(A_n) \le \mathrm{P}\left(\limsup_{n\to\infty} A_n\right).$$

Aufgabe 1.2 A collection of sets $\mathcal{A}$ is said to be a generating π-system of a σ-field $\mathcal{F}$ if $\sigma(\mathcal{A}) = \mathcal{F}$ and if $\mathcal{A}$ is a π-system[2].

(1) Show that $\mathcal{A} = \{[0, a] \colon a \in [0, 1]\}$ is a π-system generating $\mathcal{B}([0, 1])$.
(2) Prove that $\mathcal{A}' = \{(-\infty, a_1] \times \cdots \times (-\infty, a_d] \colon a_1, \ldots, a_d \in \mathbb{R}\} \cup \{\mathbb{R}^d\}$ is a π-system generating the σ-algebra $\mathcal{B}(\mathbb{R}^d)$.

Aufgabe 1.3 **(Questions and operations on σ-fields)**

(1) Is the set of all open sets of $\mathbb{R}$ a σ-field?
(2) For every $n \ge 0$, define on $\mathbb{N}$ the σ-field $\mathcal{F}_n = \sigma(\{\{0\}, \{1\}, \ldots, \{n\}\})$. Show that the sequence of σ-fields $(\mathcal{F}_n, n \ge 0)$ is non-decreasing but that $\bigcup_{n\ge 0} \mathcal{F}_n$ is not a σ-field.
Hint: argue by contradiction and use the subset of even integers.
(3) We throw two coins. To model the outcome, we use the probability space $\Omega = \{00, 01, 10, 11\}$ equipped with the σ-field $\mathcal{P}(\Omega)$. Let P be the probability measure on Ω corresponding to the case where the two coins are fair and are thrown independently. Let Q be the probability measure on Ω corresponding to the case where the second coin is rigged and always gives the same result as the first one. Show that the set $\{A \in \mathcal{P}(\Omega) : \mathrm{P}(A) = \mathrm{Q}(A)\}$ is not a σ-field (this gives in particular an example of a Dynkin system which is not a σ-field).

[1] *Almost all* means "all except finitely many exceptions". An equivalent definition of the $\liminf$ is $\liminf_{n\to\infty} A_n = \{\omega \in \Omega \mid \exists n \ge 1 : \forall m \ge n : \omega \in A_m\}$.
[2] In Notizen 2 (Seite 6) we have defined that $\mathcal{A}$ is a π-system, iff it is closed under finite intersections (meaning that for every $A, B \in \mathcal{A}$ we have $A \cap B \in \mathcal{A}$).

(4) Let $(E \times F, \mathcal{A})$ be a measured space and $\pi : E \times F \longrightarrow E$ the canonical projection defined by $\pi(x, y) = x$. Is the set $\mathcal{A}_E := \{\pi(A), A \in \mathcal{A}\}$ always a σ-field?

(5) Let $(E, \mathcal{A})$ be a measurable space. Let $\mathcal{C}$ be a collection of subsets of E, and fix $B \in \sigma(\mathcal{C})$. Alexandra says: there always exists a countable collection $\mathcal{D} \subset \mathcal{C}$ such that $B \in \sigma(\mathcal{D})$. Is she correct?

Aufgabe 1.4 Let $(E, \mathcal{E}, \mu)$ be a measured space with μ finite. Let $\mathcal{A}$ be a collection of subsets such that:
(a) $E \in \mathcal{A}$ (b) if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$ (c) if $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$ (d) $\sigma(\mathcal{A}) = \mathcal{E}$.

The goal of this exercise is to show that for every $E \in \mathcal{E}$, for every $\epsilon > 0$ there exists $A \in \mathcal{A}$ such that $\mu(E \Delta A) \leq \epsilon$.[3] To this end, set $\mathcal{S} = \{E \in \mathcal{E} : \forall \epsilon > 0, \exists A \in \mathcal{A} : \mu(E \Delta A) \leq \epsilon\}$.

(1) Show that $\mathcal{S}$ is stable by finite unions.

(2) Show that $\mathcal{S}$ is a σ-field (for stability by countable unions, justify that one may assume that the events are pairwise disjoint).

Wenn du Feedback zur Serie hast, schreibe bitte in das Forum (oder eine Mail an Jakob Heiss).

[3] In Notizen 2 (Seite 24), we have visualized the *symmetric difference* $X \Delta Y = (X \cup Y) \setminus (X \cap Y) = (X \setminus Y) \cup (Y \setminus X)$. Note that intuitively $\mu(E \Delta A) \leq \epsilon$ means that A is a good approximation of E.