Wahrscheinlichkeit und Statistik

Lösungen Serie 1

Version 2 (11. März 2024: Tippfehler in der Lösung von Aufgabe 1.1(1)(b): "$-3 \leq x \leq 1$" wurde durch "$-3 \leq x < 1$" ausgebessert. Kleiner Tippfehler in Aufgabe 1.1(2) ausgebessert. Zusätzliche Erklärungen in der Lösung von Aufgabe 1.2(1). Die Notation wurde an die Vorlesung angepasst, wo Wahrscheinlichkeitsmasse typischerweise mit "$\mathrm{P}(\cdot)$" anstatt mit "$\mathbb{P}[\cdot]$" bezeichnet werden. Minimale Verbesserungen der Formulierungen und Formatierung.), Version 1 (7. März)
Bitte stellt Fragen in den Übungen und/oder im Forum.
Bitte stell sicher, dass du die Webseite `https://kahoot.it/` in der Übung am **05. März** öffnen kannst.
Freiwillige Abgabe bis **07. März 8:00**. Nachher kann selbstständig mit dieser Lösung verglichen werden.

Aufgabe 1.1 **($\liminf$ and $\limsup$ of sets)**
Let (A_n) be a sequence of events on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Recall (from measure theory) the definitions

$$\liminf_{n\to\infty} A_n := \bigcup_{n\geq 1} \bigcap_{m\geq n} A_m,$$
$$\limsup_{n\to\infty} A_n := \bigcap_{n\geq 1} \bigcup_{m\geq n} A_m.$$

Intuition: $\liminf_{n\to\infty} A_n$ consists of the elements $\omega \in \Omega$ that appear in *almost all*[1] sets A_n.
Intuition: $\liminf_{n\to\infty} A_n$ consists of the elements $\omega \in \Omega$ that appear in infinitely many sets A_n.

(1) If $\Omega = \mathbb{R}$, give the set $\limsup_{n\to\infty} A_n$ in the following three cases (please justify your answers):
   (a) $A_n = [-1/n, 3 + 1/n]$
   (b) $A_n = [-2 - (-1)^n, 2 + (-1)^{n+1})$
   (c) $A_n = p_n\mathbb{N}$, where $(p_n)_{n\geq 1}$ is the sequence of prime numbers and $p_n\mathbb{N}$ denotes the set of all multiples of p_n.

(2) Show that

$$\mathrm{P}\left(\liminf_{n\to\infty} A_n\right) \leq \liminf_{n\to\infty} \mathrm{P}(A_n) \leq \limsup_{n\to\infty} \mathrm{P}(A_n) \leq \mathrm{P}\left(\limsup_{n\to\infty} A_n\right).$$

Lösung 1.1

(1) Note that the set $\limsup_{n\to\infty} A_n$ represents the real numbers that appear in infinitely many sets A_n:

   (a) We have $\limsup_{n\to\infty} A_n = [0, 3]$.
   First solution. Observe that for every $n \geq 1$:

$$\bigcup_{m\geq n} A_m = [-1/n, 3 + 1/n].$$

   We argue by double inclusion.
   First, for every $m \geq n$ we have $[0, 3] \subset A_m$, so $[0, 3] \subset \bigcup_{m\geq n} A_m$, so $[0, 3] \subset \bigcap_{n\geq 1} \bigcup_{m\geq n} A_m$.
   Second, if $x \in \bigcap_{n\geq 1} \bigcup_{m\geq n} A_m$, then for every $n \geq 1$ we have $-1/n \leq x \leq 3 + 1/n$ and by passing to the limit as $n \to \infty$ we get $0 \leq x \leq 3$. Thus $\bigcap_{n\geq 1} \bigcup_{m\geq n} A_m \subset [0, 3]$.
   Second solution. We argue by double inclusion. Clearly if $0 \leq x \leq 3$ then x belongs to infinitely many A_n's (it actually belongs to all of them), so $[0, 3] \subset \limsup_{n\to\infty} A_n$. Also, if $x \notin [0, 3]$ that for n sufficiently large $x \notin A_n$, so x does not belong to infinitely many A_n's. Thus $x \notin \limsup_{n\to\infty} A_n$. This shows that $\limsup_{n\to\infty} A_n \subset [0, 3]$.

[1] *Almost all* means "all except finitely many exceptions". An equivalent definition of the $\liminf$ is $\liminf_{n\to\infty} A_n = \{\omega \in \Omega \mid \exists n \geq 1 : \forall m \geq n : \omega \in A_m\}$.

For the next questions, we will use the more "intuitive" approach of the second fact, based on the fact that the set $\limsup_{n\to\infty} A_n$ represents the real numbers that appear in infinitely many sets A_n.

(b) We have $\limsup_{n\to\infty} A_n = [-3,3)$. We show this equality between sets by double inclusion. Clearly, if $-3 \le x < 3$, then x belongs to infinitely many A_n's (if $-1 \le x < 3$, $x \in A_{2n+1}$ for every $n \ge 1$ and if $-3 \le x < 1$ then $x \in A_{2n}$ for every $n \ge 1$), so $[-3,3) \subset \limsup_{n\to\infty} A_n$. If $x \notin [-3,3)$ than there is no n such that $x \in A_n$, so $x \notin \limsup_{n\to\infty} A_n$. This shows that $\limsup_{n\to\infty} A_n \subset [-3,3)$.

(c) We have $\limsup_{n\to\infty} A_n = \{0\}$. We show this equality between sets by double inclusion. Clearly $0 \in p_n\mathbb{N}$ for every $n \ge 1$, so $\{0\} \subset p_n\mathbb{N}$. Conversely, if $m \in \limsup_{n\to\infty} A_n$, then m is divisible by infinitely many prime numbers, which implies $m = 0$. Thus $\limsup_{n\to\infty} A_n \subset \{0\}$.

(2) Recall from the lecture (Notizen 2 (Seite 11)) the following fact:

$$\text{if } B_n \subset B_{n+1} \text{ for all } n \ge 1, \text{then} \quad \mathrm{P}(B_n) \underset{n\to\infty}{\longrightarrow} \mathrm{P}\left(\bigcup_m B_m\right). \tag{1}$$

By (1) applied to the sequence $B_n = \bigcap_{m\ge n} A_m$ we get

$$\mathrm{P}\left(\liminf_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathrm{P}\left(\bigcap_{m\ge n} A_m\right).$$

For every $m' \ge n$ we have $\bigcap_{m\ge n} A_m \subset A_{m'}$, so $\mathrm{P}(\bigcap_{m\ge n} A_m) \le \mathrm{P}(A_{m'})$ and hence $\mathrm{P}\left(\bigcap_{m\ge n} A_m\right) \le \inf_{m\ge n} \mathrm{P}(A_m)$. Therefore,

$$\mathrm{P}\left(\liminf_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathrm{P}\left(\bigcap_{m\ge n} A_m\right) \le \lim_{n\to\infty} \inf_{m\ge n} \mathrm{P}(A_m)$$

and the right-hand side is exactly the definition of the inferior limit.

The middle inequality $\liminf_{n\to\infty} \mathrm{P}(A_n) \le \limsup_{n\to\infty} \mathrm{P}(A_n)$ of the statement is trivial.

On could show the final inequality as the first one, but we use the very useful complementation trick by applying the first inequality to the sequence (A_n^c):

$$\mathrm{P}\left(\liminf_{n\to\infty} A_n^c\right) \le \liminf_{n\to\infty} \mathrm{P}(A_n^c).$$

Then using the fact that $\liminf_{n\to\infty} A_n^c = (\limsup_{n\to\infty} A_n)^c$ and $\liminf_{n\to\infty}(-x_n) = -\limsup_{n\to\infty}(x_n)$, we thus have

$$1 - \mathrm{P}\left(\limsup_{n\to\infty} A_n\right) \le \mathrm{P}\left(\liminf_{n\to\infty} A_n^c\right) \le \liminf_{n\to\infty} \mathrm{P}(A_n^c) = \liminf_{n\to\infty}(1-\mathrm{P}(A_n)) = 1 - \limsup_{n\to\infty} \mathrm{P}(A_n),$$

and the result follows.

Aufgabe 1.2 A collection of sets $\mathcal{A}$ is said to be a generating π-system of a σ-field $\mathcal{F}$ if $\sigma(\mathcal{A}) = \mathcal{F}$ and if $\mathcal{A}$ is a π-system[2].

(1) Show that $\mathcal{A} = \{[0,a] : a \in [0,1]\}$ is a π-system generating $\mathcal{B}([0,1])$.
(2) Prove that $\mathcal{A}' = \{(-\infty, a_1] \times \cdots \times (-\infty, a_d] : a_1, \ldots, a_d \in \mathbb{R}\} \cup \{\mathbb{R}^d\}$ is a π-system generating the σ-algebra $\mathcal{B}(\mathbb{R}^d)$.

Lösung 1.2

(1) The system of sets is stable under intersections since $[0,a] \cap [0,b] = [0, a\wedge b] \in \mathcal{A}$ for $a, b \in [0,1]$. Hence $\mathcal{A}$ is a π-system on $[0,1]$.

[2] In Notizen 2 (Seite 6) we have defined that $\mathcal{A}$ is a π-system, iff it is closed under finite intersections (meaning that for every $A, B \in \mathcal{A}$ we have $A \cap B \in \mathcal{A}$).

To see that $\mathcal{A}$ generates $\mathcal{B}([0,1])$, note first that for each $a \in [0,1]$ we have $[0,a] \in \mathcal{B}([0,1])$ since $[0,a]$ is closed in $[0,1]$. Hence $\mathcal{A} \subset \mathcal{B}([0,1])$ and thus $\sigma(\mathcal{A}) \subset \mathcal{B}([0,1])$. Conversely, if $U \subset [0,1]$ is open, then it is a countable union of sets of the form $[0,a]$ or $(a,b] = [0,b] \setminus [0,a]$ with $a,b \in [0,1]$, i.e.,

$$U = \bigcup_{\substack{n\geq 0,\ :\\ [0,2^{-n}]\subset U}} [0,2^{-n}] \cup \bigcup_{\substack{n\geq 0,\ k\in 2^{-n}\mathbb{Z}:\\ k+(-2^{-n},0]\subset U}} (k+(-2^{-n},0]) \in \sigma(\mathcal{A}).$$

This yields the reverse inclusion $\mathcal{B}([0,1]) \subset \sigma(\mathcal{A})$.

(2) The system is stable under intersections since for $a_1,\dots,a_d,b_1,\dots,b_d \in \mathbb{R}$ we have

$$\begin{aligned}&((-\infty,a_1]\times\cdots\times(-\infty,a_d]) \cap ((-\infty,b_1]\times\cdots\times(-\infty,b_d])\\ &\quad = (-\infty,a_1\wedge b_1]\times\cdots\times(-\infty,a_d\wedge b_d] \in \mathcal{A}'\ .\end{aligned}$$

Hence $\mathcal{A}'$ is a π-system on $\mathbb{R}^d$.
To see that $\mathcal{A}'$ generates $\mathcal{B}(\mathbb{R}^d)$, note that

$$(-\infty,a_1]\times\cdots\times(-\infty,a_d] = \bigcap_{n\geq 1}(-\infty,a_1+1/n)\times\cdots\times(-\infty,a_d+1/n)$$

and that all the sets in the intersection are open. This implies that $\mathcal{A}' \subset \mathcal{B}(\mathbb{R}^d)$. Conversely, fix an open set $U \subset \mathbb{R}^d$. First of all, one uses the definition of $\mathcal{A}'$ to see that $(a_1,b_1]\times\cdots\times(a_d,b_d] \in \sigma(\mathcal{A}')$ whenever $a,b \in \mathbb{R}^d$. Hence

$$U = \bigcup_{\substack{n\geq 0,\ k\in 2^{-n}\mathbb{Z}^d:\\ k+(-2^{-n},0]^d\subset U}} (k+(-2^{-n},0]^d) \in \sigma(\mathcal{A}')$$

Since the open set U was arbitrary, we get $\mathcal{B}(\mathbb{R}^d) \subset \sigma(\mathcal{A}')$ as required.

Aufgabe 1.3 (Questions and operations on σ-fields)

(1) Is the set of all open sets of $\mathbb{R}$ a σ-field?

(2) For every $n \geq 0$, define on $\mathbb{N}$ the σ-field $\mathcal{F}_n = \sigma(\{\{0\},\{1\},\dots,\{n\}\})$. Show that the sequence of σ-fields $(\mathcal{F}_n, n \geq 0)$ is non-decreasing but that $\bigcup_{n\geq 0}\mathcal{F}_n$ is not a σ-field.
Hint: argue by contradiction and use the subset of even integers.

(3) We throw two coins. To model the outcome, we use the probability space $\Omega = \{00,01,10,11\}$ equipped with the σ-field $\mathcal{P}(\Omega)$. Let P be the probability measure on Ω corresponding to the case where the two coins are fair and are thrown independently. Let Q be the probability measure on Ω corresponding to the case where the second coin is rigged and always gives the same result as the first one. Show that the set $\{A \in \mathcal{P}(\Omega) : \mathrm{P}(A) = \mathrm{Q}(A)\}$ is not a σ-field (this gives in particular an example of a Dynkin system which is not a σ-field).

(4) Let $(E \times F, \mathcal{A})$ be a measured space and $\pi : E \times F \longrightarrow E$ the canonical projection defined by $\pi(x,y) = x$. Is the set $\mathcal{A}_E := \{\pi(A), A \in \mathcal{A}\}$ always a σ-field?

(5) Let $(E,\mathcal{A})$ be a measurable space. Let $\mathcal{C}$ be a collection of subsets of E, and fix $B \in \sigma(\mathcal{C})$. Alexandra says: there always exists a countable collection $\mathcal{D} \subset \mathcal{C}$ such that $B \in \sigma(\mathcal{D})$. Is she correct?

Lösung 1.3

(1) No, because the complement of $(-\infty,0)$ is not open.

(2) The fact that the sets are non-decreasing comes from the fact that if $\mathcal{A} \subset \mathcal{B}$, then $\sigma(\mathcal{A}) \subset \sigma(\mathcal{B})$. Set

$$\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n,$$

and assume that $\mathcal{F}$ is a σ-field. We have

$$\{2n\} \in \mathcal{F}_{2n} \subset \mathcal{F} \quad \text{and} \quad 2\mathbb{N} = \bigcup_{n\geq 0}\{2n\}.$$

Therefore, $2\mathbb{N} \in \mathcal{F}$ *i.e.* there exists $n_0 \in \mathbb{N}$ such that $2\mathbb{N} \in \mathcal{F}_{n_0}$. But the only sets of $\mathcal{F}_{n_0}$ which have infinitely many elements are of the form $\mathbb{N} \setminus A$, where A is a subset of $\{0,1,\dots,n_0\}$. Indeed, elements

of $\mathcal{F}_n$ are of the form A or $A \cup \{n+1, n+2, \dots\}$ with $A \subset \{0, 1, \dots, n\}$ (to see it, one checks that such elements form a σ-field containing $\{1\}, \{2\}, \dots, \{n\}$, and that conversely any σ-field containing $\{1\}, \{2\}, \dots, \{n\}$ contains all elements of this type).
This gives a contradiction.

(3) We equip Ω with the σ-field $\mathcal{P}(\Omega)$. The probability measures P and Q are given by

$$\mathrm{P}(00) = \mathrm{P}(01) = \mathrm{P}(10) = \mathrm{P}(11) = \frac{1}{4}, \qquad \mathrm{Q}(00) = \mathrm{Q}(11) = \frac{1}{2}.$$

Then $\{A \in \Omega : \mathrm{P}(A) = \mathrm{Q}(A)\}$ is equal to

$$\{\{00, 01\}, \{00, 10\}, \{11, 01\}, \{11, 10\}, \varnothing, \Omega\},$$

which is not a σ-field since it is not stable by intersections.

(4) Take $E = F = \{0, 1\}$ and consider the σ-field $\mathcal{F}$ generated by the element $(0,0) \in E \times F$. It is clear that

$$\mathcal{F} = \{\varnothing, E \times F, \{(0,0)\}, E \times F \backslash \{(0,0)\}\}.$$

We check that $\mathcal{F}_E = \{\varnothing, \{0\}, E\}$, which is not a σ-field.

(5) Alexandra is correct. Indeed, set

$$\mathcal{G} = \{B \in \sigma(\mathcal{C})\,;\, \exists \mathcal{D} \subset \mathcal{C} \text{ countable such that } B \in \sigma(\mathcal{D})\}.$$

Let us show that $\mathcal{G}$ is a σ-field.
It is clear that $E \in \mathcal{G}$.
If $A \in \mathcal{G}$, then there exists $\mathcal{D} \subset \mathcal{C}$ countable such that $A \in \sigma(\mathcal{D})$, so $A^c \in \sigma(\mathcal{D})$: we have $A^c \in \mathcal{G}$.
If $(A_n) \subset \mathcal{G}$, then for every n there exists $\mathcal{D}_n \subset \mathcal{C}$ countable such that $A_n \in \sigma(\mathcal{D}_n)$, so $\cup_n A_n \in \sigma(\mathcal{D})$, where $\mathcal{D} := \cup_n D_n \subset \mathcal{C}$ is countable (being a countable union of countable sets): we have $\cup_n A_n \in \mathcal{G}$.
We conclude that $\mathcal{G}$ is a σ-field.
But $\mathcal{C} \subset \mathcal{G}$, which implies that $\sigma(\mathcal{C}) \subset \sigma(\mathcal{G}) = \mathcal{G} \subset \sigma(\mathcal{C})$, hence the result.
Remark. One could be led to think that $\sigma(\mathcal{C})$ can be explicitly constructed from $\mathcal{C}$ by adding all the countable unions of elements of $\mathcal{C}$ and of their complements, and then by iterating infinitely many times. In general, the result obtained in such a way can be strictly smaller than $\sigma(\mathcal{C})$.

Aufgabe 1.4 Let $(E, \mathcal{E}, \mu)$ be a measured space with μ finite. Let $\mathcal{A}$ be a collection of subsets such that:
(a) $E \in \mathcal{A}$ (b) if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$ (c) if $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$ (d) $\sigma(\mathcal{A}) = \mathcal{E}$.
The goal of this exercise is to show that for every $E \in \mathcal{E}$, for every $\epsilon > 0$ there exists $A \in \mathcal{A}$ such that $\mu(E \Delta A) \leq \epsilon$.[3] To this end, set $\mathcal{S} = \{E \in \mathcal{E} : \forall \epsilon > 0, \exists A \in \mathcal{A} : \mu(E \Delta A) \leq \epsilon\}$.

(1) Show that $\mathcal{S}$ is stable by finite unions.

(2) Show that $\mathcal{S}$ is a σ-field (for stability by countable unions, justify that one may assume that the events are pairwise disjoint).

Lösung 1.4

(1) Take $A, B \in \mathcal{S}$. Fix $\epsilon > 0$ and consider $A', B' \in \mathcal{A}$ such that $\mu(A \Delta A') \leq \epsilon$ and $\mu(B \Delta B') \leq \epsilon$. But $(A \cup B)\Delta(A' \cup B') \subset (A \Delta A') \cup (B \Delta B')$, so $\mu((A \cup B)\Delta(A' \cup B')) \leq 2\epsilon$.

(2) First of all, $E \in \mathcal{S}$ since $E \in \mathcal{A}$.
The class $\mathcal{S}$ is stable by complementation since $E \Delta A = E^c \Delta A^c$ and $\mathcal{A}$ is stable by complementation.
Let us show that $\mathcal{S}$ is stable by countable union. Thanks to question (1), we can consider a sequence (A_n) of disjoint elements of $\mathcal{S}$. Fix $\epsilon > 0$ and consider $A'_n \in \mathcal{A}$ such that $\mu(A_n \Delta A'_n) \leq \epsilon 2^{-n}$ for every $n \geq 1$. There exists N such that $\mu\left(\bigcup_{j \geq N+1} A_j\right) \leq \varepsilon/2$ (since the (A_i) are disjoint). Set $A' := \bigcup_{j=1}^{N} A'_j \in \mathcal{A}$. Since

$$\left(\bigcup_{k \geq 1} A_k\right) \Delta A' \subset \bigcup_{j=1}^{N} (A_j \Delta A'_j) \cup \bigcup_{k \geq N+1} A_k,$$

we are done.

[3] In Notizen 2 (Seite 24), we have visualized the *symmetric difference* $X \Delta Y = (X \cup Y) \backslash (X \cap Y) = (X \setminus Y) \cup (Y \setminus X)$. Note that intuitively $\mu(E \Delta A) \leq \epsilon$ means that A is a good approximation of E.

Wenn du Feedback zur Serie hast, schreibe bitte in das Forum (oder eine Mail an Jakob Heiss).