TENSOR PRODUCTS.

Remark. In this problem sheet, R is considered to be a commutative ring with 1.

Problem 1. a) Compute $\mathbb{Z}/(n\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/(m\mathbb{Z})$.

b) Let M be an R-module and $J \subset R$ an ideal. Construct an isomorphism of R-modules

$$(R/J) \otimes_R M \cong M/JM.$$

Problem 2. Let N be an R-module, and let

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of R-modules.

a) [TENSOR PRODUCT IS RIGHT EXACT] Show that the sequence

$$M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$$

is exact.

b) [Hom_R-FUNCTORS ARE LEFT EXACT] Let $\mathrm{Hom}_R(N, M)$ be the set of R-homomorphisms from N to M. Endow $\mathrm{Hom}_R(N, M)$ with an R-module structure, and show that the sequences

$$0 \longrightarrow \mathrm{Hom}_R(N, M') \longrightarrow \mathrm{Hom}_R(N, M) \longrightarrow \mathrm{Hom}_R(N, M'')$$

$$0 \longrightarrow \mathrm{Hom}_R(M'', N) \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(M', N)$$

are exact.

c) For each of the three exact sequences in a) and b), find examples of R, M, M', M'', N such that the exact sequence cannot be extended to a short exact sequence. Here, a *short exact sequence* is an exact sequence $0 \to * \to * \to * \to 0$.

Problem 3. [Hom_R AND $\otimes_R$ ARE ADJOINT] **a)** Let M, N and K be given R-modules. Show that there exists an isomorphism

$$\mathrm{Hom}_R(M \otimes_R N, K) \cong \mathrm{Hom}_R\big(M, \mathrm{Hom}_R(N, K)\big)$$

that is natural in M, N and K. Naturality in M, for example, means that whenever we have a morphism $f\colon M \to M'$ there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(M \otimes_R N, K) & \xrightarrow{\cong} & \mathrm{Hom}_R\big(M, \mathrm{Hom}_R(N, K)\big) \\ {\scriptstyle f^*}\uparrow & & {\scriptstyle f^*}\uparrow \\ \mathrm{Hom}_R(M' \otimes_R N, K) & \xrightarrow{\cong} & \mathrm{Hom}_R\big(M', \mathrm{Hom}_R(N, K)\big). \end{array}$$

b*) Use this to solve **Problem 1b)** and **Problem 2ab)** one more time.

Problem 4. a) Let N be a free R-module, and let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of R-modules. Show that the sequence

$$0 \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$$

is exact.

b) For a short exact sequence of abelian groups

$$0 \to F \to G \to H \to 0$$

construct a long exact sequence of homology groups with coefficients

$$\cdots \to H_i(X;F) \to H_i(X;G) \to H_i(X;H) \to H_{i-1}(X;F) \to \cdots$$

c) Show that there are short exact sequences

$$0 \to H_i(X;\mathbb{Z})/\big(nH_i(X;\mathbb{Z})\big) \to H_i\big(X;\mathbb{Z}/(n\mathbb{Z})\big) \to \mathrm{Tors}_n\big(H_{i-1}(X;\mathbb{Z})\big) \to 0$$

where $\mathrm{Tors}_n(G)$ is the kernel of the map $G \xrightarrow{n\cdot} G$.

Problem 5*. [Hilbert's third problem] **a)** Dissect[1] a regular triangle of area 1 into a square of area 1.

Let P be a convex polytope with edge lengths l_i and edge dihedral angles α_i. Define *Dehn's invariant* of P

$$\mathrm{Dehn}(P) \coloneqq \sum l_i \otimes \alpha_i \;\in\; \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}/\pi\mathbb{Q}.$$

b) Show that if P is split by a plane into P_1 and P_2, then $\mathrm{Dehn}(P) = \mathrm{Dehn}(P_1) + \mathrm{Dehn}(P_2)$. Deduce that if P can be dissected into P' then $\mathrm{Dehn}(P) = \mathrm{Dehn}(P')$.

c) Express $\cos(n\varphi)$ as a polynomial of $\cos(\varphi)$. What is its leading coefficient? Deduce that $\arccos(\frac{1}{3})$ is not a rational multiple of π.

d) Prove that a cube of volume 1 cannot be dissected into a regular tetrahedron of the same volume.

[1] cut the triangle along straight lines into finitely many pieces to assemble a square