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Algebraic Topology II

Problem Sheet 2
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Projective spaces, Homology with Coefficients, and the Borsuk-Ulam Theorem.

Definition. *Complex projective n-space* $\mathbb{C}P^n$ is $\mathbb{C}^{n+1} \setminus \{0\}$ modulo the equivalence relation $z \sim \lambda z$ for all $z \in \mathbb{C}^{n+1} \setminus \{0\}, \lambda \in \mathbb{C} \setminus \{0\}$.

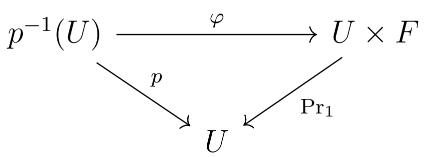
Problem 1. a) Consider S^{2n+1} to be a subset of $\mathbb{C}^{n+1}$, and let $h\colon S^{2n+1} \to \mathbb{C}P^n$ be the map given by

$$h\colon (z_0, \dots, z_n) \mapsto [z_0 : \cdots : z_n].$$

Show that[1] $\mathbb{C}P^{n+1} \cong \mathbb{C}P^n \cup_h D^{2n+2}$.

b) Deduce a CW-structure on $\mathbb{C}P^n$ and compute homology $H_\bullet(\mathbb{C}P^n; M)$, where M is an abelian group.

Definition (Fibre bundle)**.** A *fibre bundle* is a quadruple (E, p, B, F), where E, B and F are topological spaces and $p\colon E \to B$ is a continuous surjective map such that for every point $x \in B$ there exists a neighbourhood U of x and a homeomorphism $\varphi\colon p^{-1}(U) \xrightarrow{\cong} U \times F$ making the diagram below commute.



The space B is called the *base space* of the fibre bundle, E the *total space* and F the *fibre*. The map p is called the *projection map*.

c*) Prove that $h\colon S^{2n+1} \to \mathbb{C}P^n$ is a fibre bundle with the fibre S^1.

Definition (Covering)**.** A fibre bundle with discrete fibre is called a *covering*.

Problem 2. a) Construct a covering $h_{\mathbb{R}}\colon S^n \to \mathbb{R}P^n$.

b) Prove that[1] $\mathbb{R}P^{n+1} \cong \mathbb{R}P^n \cup_{h_{\mathbb{R}}} D^{n+1}$.

c) Deduce that $\mathbb{R}P^n/\mathbb{R}P^{n-1} \cong S^n$ and compute the degree of the composition

$$S^n \xrightarrow{h_{\mathbb{R}}} \mathbb{R}P^n \to \mathbb{R}P^n/\mathbb{R}P^{n-1} \xrightarrow{\cong} S^n.$$

d) Describe the cellular chain complex $C_\bullet^{CW}(\mathbb{R}P^n; M)$

$$\cdots \xrightarrow{d} C_k^{CW}(\mathbb{R}P^n; M) \xrightarrow{d} C_{k-1}^{CW}(\mathbb{R}P^n; M) \xrightarrow{d} \cdots \xrightarrow{d} C_0^{CW}(\mathbb{R}P^n; M),$$

and compute the homology $H_\bullet(\mathbb{R}P^n; M)$ for $M = \mathbb{Z}$ and $\mathbb{F}_2$.

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[1] Recall, that given a map $f\colon A \to Y$ from a subspace $A \subseteq X$, one can define $Y \cup_f X$ to be the quotient of $Y \sqcup X$ by the equivalence relation $x \sim f(x)$ for $x \in A$.

Problem 3. Let $n > m > 1$. **a)** Prove that for any map $f\colon \mathbb{R}P^n \to \mathbb{R}P^m$ the induced map $f_*\colon H_1(\mathbb{R}P^n;\mathbb{F}_2) \to H_1(\mathbb{R}P^m;\mathbb{F}_2)$ is trivial.

Hint: consider the map between two Gysin sequences induced by a lift of f to $S^n \to S^m$.

b) Deduce that $\mathbb{R}P^m$ is not a retract of $\mathbb{R}P^n$.

Problem 4. **a)** Construct a map $f\colon \mathbb{R}P^2 \to S^2$ such that

$$f_*\colon H_2(\mathbb{R}P^2;\mathbb{F}_2) \to H_2(S^2;\mathbb{F}_2)$$

is non-trivial.

b) Deduce that there exists a pair of continuous maps with the same domain and codomain that induce the same homomorphisms on integral homology, but different homomorphisms on homology with $\mathbb{F}_2$ coefficients.

c*) Show that there exists no natural choice of splitting maps in the universal coefficient theorem.

Problem 5. Does the Borsuk-Ulam theorem hold for the torus? Namely, for every map $f\colon S^1 \times S^1 \to \mathbb{R}^2$ there exists $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$?

Problem 6. [Lusternik-Schnirelman theorem] Prove that if the sphere S^n is covered by $n + 1$ closed sets, then one of the sets contains a pair of antipodal points.