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Algebraic Topology II

Problem Sheet 3
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Cohomology and the Universal Coefficient Theorem

In your solutions, you may use (1)–(4) of Prop 4.14 (which were proven in the lecture). If you use (5)–(8) of Prop 4.14 (which were stated without proof in the lecture), you should prove them yourself.

Problem 1. Compute $\operatorname{Tor}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ for all $m, n \geq 0$.

Problem 2. Prove that for any abelian groups A, B **a)** $\operatorname{Tor}(A, B)$ is a torsion group; **b)** $\operatorname{Tor}(A, \mathbb{Q}/\mathbb{Z})$ is isomorphic to the torsion subgroup $T(A)$ of A.

Problem 3. Let $f\colon X \to Y$ be a continuous map.

a) Show that if $f_*\colon H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z})$ is an isomorphism for all n, then $f_*\colon H_n(X;M) \to H_n(Y;M)$ is an isomorphism for all n and all abelian groups M.

b) Prove that $f_*\colon H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z})$ is an isomorphism for all n if and only if f induces isomorphisms on homology with $\mathbb{Q}$ and $\mathbb{F}_p$ coefficients for all primes p.

Hint: use the long exact sequence for Tor to show that if $A \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ and $\operatorname{Tor}(A, \mathbb{F}_p) = 0$ for all primes p, then $A = 0$.

Problem 4. a) Show that $H_n(X;\mathbb{Q}) \cong H_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

b) Prove that if $H_n(X;\mathbb{Z})$ and $H_{n-1}(X;\mathbb{Z})$ are finitely generated, then for a prime p, $H_n(X;\mathbb{F}_p)$ consists of

- an $\mathbb{F}_p$ summand for each $\mathbb{Z}$ summand of $H_n(X;\mathbb{Z})$;
- an $\mathbb{F}_p$ summand for each $\mathbb{Z}/p^k\mathbb{Z}$ summand of $H_n(X;\mathbb{Z})$, $k \geqslant 1$;
- an $\mathbb{F}_p$ summand for each $\mathbb{Z}/p^k\mathbb{Z}$ summand of $H_{n-1}(X;\mathbb{Z})$, $k \geqslant 1$.

Problem 5*. Construct a free resolution of $\mathbb{Z}/2\mathbb{Z}$ as $\mathbb{Z}/4\mathbb{Z}$-module and compute

$$\operatorname{Tor}_n^{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}), \qquad \operatorname{Ext}^n_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$$

for all $n \geq 0$.

Problem 6. For a path $\sigma\colon [0,1] \to S^1$, denote by $\widetilde{\sigma}\colon [0,1] \to \mathbb{R}$ any lift to the covering $\mathbb{R} \to S^1, t \mapsto e^{2\pi i t}$. In other words, $e^{2\pi i \widetilde{\sigma}(t)} = \sigma(t)$ for all $t \in [0,1]$.

a) Prove that sending σ to $\widetilde{\sigma}(1) - \widetilde{\sigma}(0)$ gives a well-defined 1-cochain with $\mathbb{R}$ coefficients of S^1, i.e. an element of $C^1(S^1;\mathbb{R})$; and that in fact, it is a 1-cocycle generating $H^1(S^1;\mathbb{R}) \cong \mathbb{R}$.

b) Similarly, prove that sending σ to $\lfloor\widetilde{\sigma}(1)\rfloor - \lfloor\widetilde{\sigma}(0)\rfloor$ defines a 1-cocycle generating $H^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$. Here, $\lfloor x\rfloor$ of a real number x is the *floor* of x, i.e. the largest integer less than or equal to x.

Problem 7. Show that $H^1(X;\mathbb{Z})$ has no torsion.

Problem 8. Compute cellular cohomology with $\mathbb{Z}$ and $\mathbb{F}_2$ coefficients of **a)** the n-dimensional torus $T^n = (S^1)^{\times n}$; **b)** the Klein bottle; **c)** the real projective space $\mathbb{R}P^n$.
Check in a), b), c) that your result is consistent with the universal coefficient theorem of cohomology.

Problem 9*. Describe the connecting homomorphism

$$\beta\colon H^\bullet(\mathbb{R}P^n;\mathbb{F}_2) \to H^{\bullet+1}(\mathbb{R}P^n;\mathbb{F}_2)$$

associated with the coefficient exact sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$.