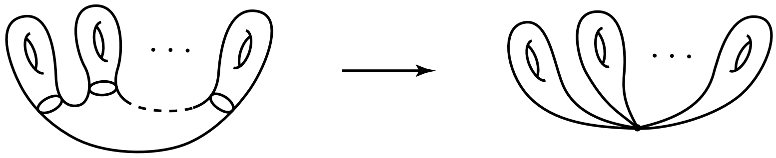
COHOMOLOGY AND CUP-PRODUCT

Problem 1. Using the evaluation map $\mathrm{ev}\colon H^n(X;M) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(X), M)$ and its relative version, find the relation between the connecting homomorphisms $\partial\colon H_{n+1}(X,A;M) \to H_n(A;M)$ and $\delta\colon H^n(A;M) \to H^{n+1}(X,A;M)$.

Problem 2. Using Δ-complexes and simplicial (co-)homology, compute the cohomology rings with coefficients in $\mathbb{Z}$ and in $\mathbb{F}_2$ of **a)** the Klein bottle K; **b)** the projective plane $\mathbb{R}P^2$.

Problem 3. Compute the cohomology ring with coefficients in $\mathbb{Z}$ for the closed orientable surface M_g of genus $g \geqslant 2$, using the quotient map from M_g to the wedge sum of g tori shown below (figure from Hatcher).



Problem 4. Show that $\mathbb{C}P^2$ and $S^2 \vee S^4$ are not homotopy equivalent.

Definition (Lusternik–Schnirelmann category)**.** The Lusternik–Schnirelmann category of a topological space X is the smallest integer $k \geqslant 1$ such that there exists an open cover $\{U_i\}_{i=1}^k$ of X with nullhomotopic inclusions $U_i \hookrightarrow X$.

Problem 5. a) Show that the Lusternik–Schnirelmann category of X is bounded from below by the maximal length of a non-zero product of positive degree elements in $H^\bullet(X)$. In other words, if $\alpha_1 \smile \cdots \smile \alpha_k \neq 0$ for some $\alpha_i \in H^{n_i}(X)$ with $n_i \geqslant 1$, then the Lusternik–Schnirelmann category of X is at least $k+1$.

Compute the Lusternik–Schnirelmann category of **b)** the sphere S^n; **c)** the projective space $\mathbb{C}P^n$; **d)** the n-dimensional torus T^n.

HINT FOR **b)**. Let $p\colon \mathbb{R}^n \xrightarrow{(t_1,\ldots,t_n)\mapsto(e^{2\pi\sqrt{-1}t_1},\ldots,e^{2\pi\sqrt{-1}t_n})} T^n$ be the universal covering. Choose $a_0 < \cdots < a_n \in [0,1]$. Show that the charts $U_k \coloneqq p(\{(t_1,\ldots,t_n) \in \mathbb{R}^n \mid a_k < t_i < a_k + 1, \text{for all } i\})$ for $0 \leqslant k \leqslant n$ give the required covering of T^n.