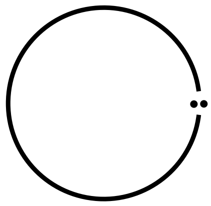
Manifolds and Orientations

Problem 1. Show that the following spaces are manifolds:
a) the Möbius band; **b)** the n-dimensional torus T^n; **c)** the real projective space $\mathbb{R}P^n$; **d)** the complex projective space $\mathbb{C}P^n$; **e)** the general linear group $GL_n(\mathbb{R})$; **f*)** the special orthogonal group SO_n.

Problem 2. Consider X the "circle with two 1s". That is, X is $S^1 \times \{0,1\}$ modulo the equivalence relation $\sim$, where $(e^{it}, a) \sim (e^{is}, b)$ iff $e^{it} = e^{is} \neq 1$, or $e^{it} = e^{is} = 1$ and $a = b$. So X resembles S^1, but has "two 1s".



a) Show that X is locally Euclidean, second countable, connected, compact, satisfies $H_1(X, X \setminus x) \cong H_1(S^1)$ for all $x \in X$, and is orientable, but not Hausdorff.

b) Compute $H_1(X) = \mathbb{Z}^2$.

c) Pinpoint precisely where our proof that $H_1(X) \cong \mathbb{Z}$ fails for X.

Problem 3. Let M be an orientable manifold with a properly discontinuous action of a group G by orientation preserving homeomorphisms. Show that M/G is orientable.

Problem 4. Let M be a connected topological manifold. Show that M is orientable if $\pi_1(M)$ has no subgroup of index 2.

Problem 5. Show that the cap product turns the total homology $H_\bullet(X; R) = \bigoplus_n H_n(X; R)$ into a (graded) right-module over the cohomology ring $H^\bullet(X; R)$.

Problem 6. **a)** Let M be a closed connected oriented manifold such that there is an isomorphism of cohomology groups $H^\bullet(M; \mathbb{Q}) \cong H^\bullet(\mathbb{C}P^2; \mathbb{Q})$. Is it true that there is an isomorphism of cohomology rings?

b) Same question for an isomorphism $H^\bullet(M; \mathbb{Q}) \cong H^\bullet(\mathbb{C}P^3; \mathbb{Q})$.

Problem 7. Is there a continuous map $\mathbb{C}P^3 \to \mathbb{C}P^3$ of degree **a)** 9; **b)** 8?

Problem 8. Let M be a closed simply connected 3-dimensional manifold. Show that M is **a*)** homotopy equivalent to S^3; **b***)** homeomorphic to S^3.

Problem 9. **a)** Let $f\colon M \to N$ be a map between connected closed orientable n-manifolds. Suppose there is a ball $B \subseteq N$ such that $f^{-1}(B)$ is the disjoint union of balls B_i each mapped homeomorphically by f onto B. Show that the degree of f is $\sum_i \epsilon_i$ where ϵ_i is ± 1 according to whether $f\colon B_i \to B$ preserves or reverses local orientations induced from given fundamental classes $[M]$ and $[N]$.

b) Let $f\colon X \to Y$ be an n-sheeted covering of closed connected orientable manifolds. Show that $\deg f = \pm n$.

c) Let $f\colon X \to Y$ be a degree 1 map of closed connected orientable manifolds. Show that $f_*\colon \pi_1(X) \to \pi_1(Y)$ is surjective.

d) Let Σ_g be the orientable closed connected surface of genus $g \geqslant 0$. Show that there exists a continuous map $f\colon \Sigma_g \to \Sigma_h$ of degree 1 if and only if $g \geqslant h$.