Dr. Lukas Lewark   Algebraic Topology II   Solutions Sheet 1
ETH Zürich   Spring, 2024

Problem 1

by Naomi Rosenberg

a).

Version 1: Explicitly constructing maps.

Claim 1. *$\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/d\mathbb{Z}$, where $d = gcd(m, n)$.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \arrow[r, dashed, "f"] \arrow[dr, "\beta"] & \mathbb{Z}/d\mathbb{Z} \\
& \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}, \arrow[u, "\alpha"]
\end{tikzcd}$$

where

$\beta : \mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}\times\mathbb{Z}/d\mathbb{Z}$, $(k_1+m\mathbb{Z}, k_2+n\mathbb{Z}) \mapsto (k_1+d\mathbb{Z}, k_2+d\mathbb{Z})$, and

$$\alpha : \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z},\ (k_1 + d\mathbb{Z}, k_2 + d\mathbb{Z}) \mapsto k_1k_2 + d\mathbb{Z}.$$

Here, we used that β is well-defined since d is a divisor of both m, and n. Note that β is linear and α is bilinear, thus the composition of α and β is bilinear. With this in mind, define

$$f = \alpha \circ \beta : \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}, (k_1 + m\mathbb{Z}, k_2 + n\mathbb{Z}) \mapsto k_1k_2 + d\mathbb{Z}.$$

The map f is surjective as a composition of two surjective maps and bilinear. Indeed,

- $(k_1 + k_1' + m\mathbb{Z}, k_2 + n\mathbb{Z}) \mapsto (k_1 + k_1')k_2 + d\mathbb{Z} = k_1k_2 + d\mathbb{Z} + k_1'k_2 + d\mathbb{Z} = f(k_1+m\mathbb{Z}, k_2+n\mathbb{Z})+f(k_1'+m\mathbb{Z}, k_2+n\mathbb{Z})$ and similar for the right argument, and
- $(\lambda k_1+m\mathbb{Z}, k_2+n\mathbb{Z}) \mapsto (\lambda k_1)k_2+d\mathbb{Z} = \lambda(k_1k_2)+d\mathbb{Z} = \lambda f(k_1+m\mathbb{Z}, k_2+n\mathbb{Z})$, and analogously if λ appears on the right side.

Thus, by the universal property of the tensor product, there exists a (unique) homomorphism $\varphi : \mathbb{Z}/m\mathbb{Z}\otimes\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$, such that $f = \varphi \circ \mu$ (see figure below).

$$\begin{tikzcd}
\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \arrow[r, "f"] \arrow[d, "\mu"] & \mathbb{Z}/d\mathbb{Z} \\
\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}. \arrow[ur, dashed, "\varphi"]
\end{tikzcd}$$

It is now sufficient to show that φ is an isomorphism. In order to prove this, we are going to construct its inverse.
Note that $\mathbb{Z}/d\mathbb{Z} = \langle [1] \rangle$, where we denote by $[1]$ the equivalence class $1 + d\mathbb{Z}$ of 1 in $\mathbb{Z}/d\mathbb{Z}$.

Define

$$\psi : \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z},\ [1] \mapsto [1] \otimes [1].$$

We need to check that ψ sends $[0] = [d]$ to $0 \in \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$. For this, we need that $\mathbb{Z}$ is a principal ideal domain as this implies the existence of unique elements $u, v \in \mathbb{Z}$ such that $d = um + vn$. Thus, $\psi([0]) = \psi([d]) = [d]([1] \otimes [1]) = [um+vn]([1]\otimes[1]) = [um]\otimes[1]+[1]\otimes[vn] = [0]\otimes[1]+[1]\otimes[0] = [0] \in \mathbb{Z}/m\mathbb{Z}\otimes\mathbb{Z}/n\mathbb{Z}$.
Finally, we show that φ and ψ are inverse to each other: For any $[k] \in \mathbb{Z}/d\mathbb{Z}$ and for every generator $[a] \otimes [b]$ of $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$, the following holds:

- $\varphi \circ \psi([k]) = \varphi([k]([1] \otimes [1])) = \varphi([k] \otimes [1]) = [k][1] = [k]$, and
- $\psi \circ \varphi([a] \otimes [b]) = \varphi([a][b]) = [a][b]([1] \otimes [1]) = [a] \otimes [b]$.

Thus, ψ is indeed the inverse of φ and therefore, we can conclude that φ is an isomorphism.

□

Version 2: Using Problem 1.b). By Problem 1.b), for every ideal $J \subseteq R$, there exists an isomorphism $(R/J) \otimes M \to M/JM$ with $(r + J) \otimes x \mapsto rx + JM$. Thus, setting $R = \mathbb{Z}$, $J = n\mathbb{Z}$, and $M = \mathbb{Z}/m\mathbb{Z}$ yields

$$\begin{aligned}\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} &\overset{1.b)}{\cong} (\mathbb{Z}/m\mathbb{Z})/((n\mathbb{Z})(\mathbb{Z}/m\mathbb{Z})) \\ &\cong (\mathbb{Z}/m\mathbb{Z})/((n\mathbb{Z} + m\mathbb{Z})/m\mathbb{Z}) \\ &\cong \mathbb{Z}/(n\mathbb{Z} + m\mathbb{Z}) \\ &\cong \mathbb{Z}/gcd(m, n)\mathbb{Z}.\end{aligned}$$

b). In the following, we are going to denote the equivalence classes in R/J by $[.]$ and the equivalence classes in M/JM by $\{.\}$. Consider the following diagram:

$$\begin{tikzcd} R/J \times M \arrow[r, "f"] \arrow[d, "\mu"] & M/JM \\ R/J \otimes M, \arrow[ur, dashed, "\varphi"] & \end{tikzcd}$$

where $f : R/J \times M \to M/JM$, $([r], m) \mapsto \{rm\}$. The map f is surjective and bilinear, thus, by the universal property of the tensor product, there exists a (unique) homomorphism $\varphi : R/J \otimes M \to M/JM$. Therefore, in order to prove the statement from the exercise, it is sufficient to find an inverse of φ.
Define $\overline{\psi} : M \to R/J \otimes M$, $m \mapsto [1] \otimes m$ and note that $JM \subseteq ker(\overline{\psi})$.
Hence, there exists a homomorphism $\psi : M/JM \to R/J \otimes M$, $\{m\} \mapsto [1] \otimes m$, which is precisely the inverse of φ (because $\varphi \circ \psi(\{m\}) = \varphi([1] \otimes m) = \{m\}$, and $\psi \circ \varphi(([r], m)) = \psi(\{rm\}) = [1] \otimes rm = [r] \otimes m$) and consequently, φ is an isomorphism.

Exercise 2.

Let $(\star)$ $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ be a SES of R-mod's

Let N be an R-module.

(a) Want to show that the induced sequence

$$M' \otimes_R N \xrightarrow{\alpha \otimes \mathbb{1}} M \otimes_R N \xrightarrow{\beta \otimes \mathbb{1}} M'' \otimes N \longrightarrow 0$$

is exact.

Exactness is equavalent to having an isomorphism

$(\bigstar)$ $$\frac{M \otimes_R N}{\mathrm{Im}(\alpha \otimes \mathbb{1})} \xrightarrow{\cong} M'' \otimes N \quad \text{induced by } \beta \otimes \mathbb{1}.$$

(1) Note that $(\beta \otimes \mathbb{1}) \circ (\alpha \otimes \mathbb{1}) = \beta \circ \alpha \otimes \mathbb{1} = 0$ since $\beta \circ \alpha = 0$.

$\Longrightarrow$ $M \otimes N \xrightarrow{\beta \otimes \mathbb{1}} M'' \otimes N$ factors through $\dfrac{M \otimes_R N}{\mathrm{Im}(\alpha \otimes \mathbb{1})}$

(2) Construct inverse to $(\bigstar)$. By the universal property, we need a bilinear map $M'' \times N \xrightarrow{\varphi} \dfrac{M \otimes_R N}{\mathrm{Im}(\alpha \otimes \mathbb{1})}$

for $(m'', n) \in M'' \times N$, let $m \in M$ be s.t.

$$\beta(m) = m''$$

Define $\varphi(m'', n) = m \otimes n$. It is well-defined: if $\tilde{m} \in M$ is another element s.t. $\beta(\tilde{m}) = m''$

by exactness of $(\star)$: $\tilde{m} - m = \alpha(m')$ for some $m' \in M'$

$\longrightarrow$ $\tilde{m} \otimes n - m \otimes n = (\tilde{m} - m) \otimes n = \alpha(m') \otimes n = 0$

$M'' \otimes_R N \xrightarrow{\tilde{\varphi}} \dfrac{M \otimes_R N}{\mathrm{Im}(\alpha \otimes \mathbb{1})}$ & $(\bigstar)$ are inverse to each other.

$$\tilde{\varphi} \circ (\beta \otimes \mathbb{1})(m \otimes n) = \tilde{\varphi}(\beta(m) \otimes n) = m \otimes n$$

$$\beta \otimes \mathbb{1} \circ \tilde{\varphi}(m'' \otimes n) = \beta(m) \otimes n = m'' \otimes n.$$

6 Consider first

$$0 \to \mathrm{Hom}_R(N, M') \xrightarrow{\alpha_*} \mathrm{Hom}_R(N, M) \xrightarrow{\beta_*} \mathrm{Hom}_R(N, M'')$$

(1) Note that $M' = \ker(\beta)$.

(2) Let $f \in \mathrm{Hom}_R(N, M)$ be s.t. $\beta_*(f) = 0$.

$\Rightarrow$ the composite $N \xrightarrow{f} M \xrightarrow{\beta} M''$ is trivial

$\Rightarrow$ by the universal property of the kernel $f : N \to M$ lifts to $\tilde{f} : N \to \ker(\beta) = M'$.

exactness at $\mathrm{Hom}_R(N, M)$

(3) Let $f \in \mathrm{Hom}_R(N, M')$ be s.t. $\alpha_*(f) = 0$.

$\Rightarrow$ the composite $N \xrightarrow{f} M' \xrightarrow{\alpha} M$ is trivial.

Since α is injective, f has to be trivial.

exactness at $\mathrm{Hom}_R(N, M')$

Consider $0 \to \mathrm{Hom}_R(M'', N) \xrightarrow{\beta^*} \mathrm{Hom}_R(M, N) \xrightarrow{\alpha^*} \mathrm{Hom}_R(M', N)$

(1) Let $f \in \mathrm{Hom}_R(M, N)$ be s.t. $\alpha^*(f) = 0$

$\Rightarrow$ the composite $M' \xrightarrow{\alpha} M \xrightarrow{f} N$ is trivial

$\Rightarrow$ $M \xrightarrow{f} N$ factors through

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[rr, bend left, "f"] \arrow[dr, bend right, "\beta"'] & M/\mathrm{Im}(\alpha) \arrow[r] \arrow[d, phantom, "\cong"] & N \\ & M'' \arrow[ur, "f''"'] & \end{tikzcd}$$

canonical projection

$\Rightarrow f = f'' \circ \beta = \beta^*(f'')$

exactness at $\mathrm{Hom}_R(M, N)$

(2) Let $f \in \mathrm{Hom}_R(M'', N)$ be s.t. $\beta^*(f) = 0$

$\Rightarrow$ the composite $M \xrightarrow{\beta} M'' \xrightarrow{f} N$ is trivial.

Since β is surjective, f has to be trivial.

(c) Consider the SES: $0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$. (★)

$$\Big\downarrow \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z} \to 0.$$

by (1a)

non-trivial

$\Rightarrow$ the sequence is <u>not</u> short exact.

Taking both $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \_)$ & $\mathrm{Hom}_{\mathbb{Z}}(\ \ , \mathbb{Z}/2\mathbb{Z})$ of (★) lead to

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$$

non-trivial

$\Rightarrow$ the sequence is <u>not</u> short exact.

Problem 3

by Noah Stäuble & Philip Sandt

We start by a well-celebrated relationship from commutative algebra. We have a bijection (even an R-linear isomorphism)

$$\phi : \mathfrak{Hom}_R (M \otimes_R N, K) \to \mathfrak{Bil}_R (M, N; K), \quad f \mapsto ((m,n) \mapsto f(m \otimes n)) \tag{1}$$

In order to know that this map is well-defined (i.e. that we land in the bilinear maps) we can argue via the universal property of the tensor product. For the bijectivity we provide an inverse

$$\psi : \mathfrak{Bil}_R (M, N; K) \to \mathfrak{Hom}_R (M \otimes_R N, K) \tag{2}$$

that is obtained by sending g to the map linearly extending

$$m \otimes n \mapsto g(m,n).$$

We check that ϕ is linear. Let $a \in R$, $f, g \in \mathfrak{Hom}_R (M \otimes N, K)$. Then, ϕ maps $af + g$ to the map that sends $(m,n) \in M \times N$ to $af(m \otimes n) + g(m \otimes n)$. At the same time, $a\phi(f) + \phi(g)$ is the map that sends (m,n) to $a\phi(f)(m,n) + \phi(g)(m,n) = af(m \otimes n) + g(m \otimes n)$. Moreover, $(\psi \circ \phi)(f)$ maps $m \otimes n$ to $\phi(f)(m,n) = f(m \otimes n)$ And, $(\phi \circ \psi)(h)$ maps (m,n) to $\psi(h)(m \otimes n) = h(m,n)$ which implies that ϕ, ψ are mutually inverse, using that pure tensors generate the tensor product.
We are also going to use the result from commutative algebra that states that there is an isomorphism

$$F : \mathfrak{Hom}_R (M, \mathfrak{Hom}_R (N, K)) \to \mathfrak{Bil}_R (M, N; K)$$

such that

$$F(f)(m,n) = f(m)(n)$$

with inverse

$$G : \mathfrak{Bil}_R (M, N; K) \to \mathfrak{Hom}_R (M, \mathfrak{Hom}_R (N, K))$$

such that $G(g)(m)$ is given by $n \mapsto g(m,n)$. By composing the two, we get an isomorphism

$$G \circ \phi : \mathfrak{Hom}_R (M \otimes_R N, K) \to \mathfrak{Hom}_R (M, \mathfrak{Hom}_R (N, K))$$

It remains to show that the isomorphism is natural. Following the example, we start by showing that the isomorphism is natural in M. For what follows, let M', N', K' be R-modules. We denote for any map t between functions between R-modules by t' the map that does the same as t but where the appropriate module A has been replaced by A'.

1. Let $f : M \to M'$ a homomorphism. We define $f_* : \mathfrak{Hom}_R (M' \otimes N, K) \to \mathfrak{Hom}_R (M \otimes N, K)$ by pre-composing with f in M. I.e. by linearly extending the following relation on pure tensors:

$$f_*(g)(m \otimes n) = g(f(m) \otimes n), \textit{ for } g \in \mathfrak{Hom}_R (M' \otimes N, K), m \in M, n \in N$$

Analogously, define $f_* : \mathfrak{Hom}_R(M', \mathfrak{Hom}_R(N,K)) \to \mathfrak{Hom}_R(M, \mathfrak{Hom}_R(N,K))$, by linearly extending

$$f_*(g)(m)(n) = g(f(m))(n), \text{ for } g \in \mathfrak{Hom}_R(M', \mathfrak{Hom}_R(N,K)), m \in M, n \in N$$

Naturality is then equivalent to commutativity of the following diagram.

$$\begin{array}{ccc}
\mathfrak{Hom}_R(M \otimes N, K) & \xrightarrow{G\circ\phi} & \mathfrak{Hom}_R(M, \mathfrak{Hom}_R(N,K)) \\
{\scriptstyle f_*}\uparrow & & \uparrow{\scriptstyle f_*} \\
\mathfrak{Hom}_R(M' \otimes N, K) & \xrightarrow{G'\circ\phi'} & \mathfrak{Hom}_R(M', \mathfrak{Hom}_R(N,K))
\end{array}$$

Which we will prove by showing equality on pure tensors.

I.e. we want to show, that $G \circ \phi \circ f_*(g)(m)(n) = f_* \circ G' \circ \phi'(g)(m)(n)$, for all homomorphisms $g \in \mathfrak{Hom}_R(M' \otimes N, K)$ and elements $m \in M$ and $n \in N$. Let us first consider the upper part of the diagram.

$$\begin{aligned}
G \circ \phi \circ f_*(g)(m)(n) &= G(\phi(f_*(g)))(m)(n) \\
= \phi(f_*(g))(m,n) &= f_*(g)(m \otimes n) = g(f(m) \otimes n)
\end{aligned}$$

In other words, for $m \in M$, $G \circ \phi \circ f_*(g)(m)$ is the morphism $n \mapsto (g(f(m) \otimes n))$ in $\mathfrak{Hom}_R(N,K)$. On the other hand.

$$\begin{aligned}
f_* \circ G' \circ \phi'(g)(m)(n) &= f_*(G'(\phi'(g)))(m)(n) \\
= G'(\phi'(g))(f(m))(n) &= \phi'(g)(f(m),n) = g(f(m) \otimes n)
\end{aligned}$$

resulting in the same morphism.

2. Let $f : N \to N'$ a homomorphism. Define $f_* : \mathfrak{Hom}_R(M \otimes N', K) \to \mathfrak{Hom}_R(M \otimes N, K)$ linearly extending the following relation on pure tensors:

$$f_*(g)(m \otimes n) = g(m \otimes f(n)), \text{ for } g \in \mathfrak{Hom}_R(M \otimes N', K), m \in M, n \in N$$

Analogously, define $f_* : \mathfrak{Hom}_R(M, \mathfrak{Hom}_R(N',K)) \to \mathfrak{Hom}_R(M, \mathfrak{Hom}_R(N,K))$, by linearly extending

$$f_*(g)(m)(n) = g(m)(f(n)), \text{ for } g \in \mathfrak{Hom}_R(M, \mathfrak{Hom}_R(N',K)), m \in M, n \in N$$

To prove naturality in N, consider the following diagram:

$$\begin{tikzcd}
\mathfrak{Hom}_R(M \otimes N', K') \arrow[r, "f_*"] \arrow[d, "\eta"] & \mathfrak{Hom}_R(M \otimes N, K) \arrow[d, "\eta"] \\
\mathfrak{Hom}_R(N' \otimes M, K) \arrow[r, "f_*^{(1)}"] \arrow[d, "G' \circ \phi'"] & \mathfrak{Hom}_R(N \otimes M, K) \arrow[d, "G \circ \phi"] \\
\mathfrak{Hom}_R(N', \mathfrak{Hom}_R(M, K)) \arrow[r, "f_*^{(1)}"] \arrow[d, "\psi'^{(1)}"] & \mathfrak{Hom}_R(N, \mathfrak{Hom}_R(M, K)) \arrow[d, "\psi^{(1)}"] \\
\mathfrak{Bil}_R(M, N; K) \arrow[d, "\phi'^{(2)}"] & \mathfrak{Bil}_R(M, N; K) \arrow[d, "\phi^{(2)}"] \\
\mathfrak{Hom}_R(M, \mathfrak{Hom}_R(N', K)) \arrow[r, "f_*"] & \mathfrak{Hom}_R(M, \mathfrak{Hom}_R(N, K))
\end{tikzcd}$$

Where by $\psi^{(1)} : \mathfrak{Hom}_R(N, \mathfrak{Hom}_R(M, K)) \to \mathfrak{Bil}_R(M, N; K)$, we denote the morphism, which sends the first argument of a bilinear map to the first argument of the nested $\mathfrak{Hom}_R(\bullet, \bullet)$'s - analogously $\psi^{(2)}$, $\phi^{(1)}$ and $\phi^{(2)}$, where we implicitly use $\mathfrak{Bil}_R(M, N; K) \cong \mathfrak{Bil}_R(N, M; K)$. We will show that the above diagram is commutative, by showing that all 3 squares commute.

(a) To show that the upper square commutes, note that "reversed" of the tensor products are naturally isomorphic, for commutative R. I.e. for A, B R-Modules, there exists a natural isomorphism

$$\bar{\eta} : A \otimes B \to B \otimes A$$

Since $\mathfrak{Hom}_R(\bullet, K)$ is a functor, defining $\eta := \mathfrak{Hom}_R(\bar{\eta}, K)$ makes the upper square commute.

(b) Commutativity in the middle square is exactly the statement of 1.

(c) For commutativity of the lower square, we simply check for $g : N' \to \mathfrak{Hom}_R(M, K)$, $m \in M$ and $n \in N$.

$$\begin{aligned}
f_*(\phi'^{(2)}(\psi'^{(1)}(g)))(m)(n) &= \phi'^{(2)}(\psi'^{(1)}(g))(m)(f(n)) \\
&= \psi'^{(1)}(g)(f(n), m) = g(f(n)(m) \\
&= f_*^{(1)}(g)(n)(m) = \psi^{(1)}(f_*^{(1)}(g))(m, n) \\
&= \phi^{(2)}(\psi^{(1)}(f_*^{(1)}(g)))(m)(n).
\end{aligned}$$

3. The last piece of the puzzle is naturality in K. Let $h : K \to K'$ be R-linear. We obtain

$$h^* : \mathfrak{Hom}_R(M \otimes N, K) \to \mathfrak{Hom}_R(M \otimes N, K')$$

$$q \mapsto h \circ q$$

and

$$h^* : \mathfrak{Hom}_R\left(M, \mathfrak{Hom}_R\left(N, K\right)\right) \to \mathfrak{Hom}_R\left(M, \mathfrak{Hom}_R\left(N, K'\right)\right)$$
$$t \mapsto \left(m \mapsto h \circ t(m)\right).$$

We want commutativity in the diagram beneath.

$$\begin{array}{ccc}
\mathfrak{Hom}_R\left(M \otimes N, K\right) & \xrightarrow{G\circ\phi} & \mathfrak{Hom}_R\left(M, \mathfrak{Hom}_R\left(N, K\right)\right) \\
{\scriptstyle h^*}\big\downarrow & & \big\downarrow{\scriptstyle h^*} \\
\mathfrak{Hom}_R\left(M \otimes N, K'\right) & \xrightarrow{G'\circ\phi'} & \mathfrak{Hom}_R\left(M, \mathfrak{Hom}_R\left(N, K'\right)\right)
\end{array}$$

(a) $h^* \circ G \circ \phi$: Let $q \in \mathfrak{Hom}_R\left(M \otimes N, K\right)$. Then $q_1 := (G \circ \phi)(q)$ is the map such that for all $m \in M$, $q_1(m)(n) = q(m \otimes n)$. Similarly, $h^* q_1$ is the homomorphism that satisfies for all $m \in M : (h^* q_1)(m) = \left(n \mapsto \left(h \circ q_1(m)\right)(n)\right)$. Hence the resulting element in the bottom right module is given by the map that to any $m \in M$ associates the linear map $n \mapsto h\left(q_1(m)(n)\right) = h\left(q(m \otimes n)\right)$.

(b) $G' \circ \phi' \circ h^*$: Let $q \in \mathfrak{Hom}_R\left(M \otimes N, K\right)$. Then $q_2 := h^* q$ is the map defined on pure tensors by $m \otimes n \mapsto h\left(q(m \otimes n)\right) \in K'$. We compute that $(G' \circ \phi')(q_2)$ is the map that maps $m \in M$ to the homomorphism

$$n \mapsto q_2(m \otimes n) = h\left(q(m \otimes n)\right).$$

We showed that the above diagram commutes, which concludes the proof.

PROBLEM 4

by Sina Keller and Tristan Lovsin

4a). Let's restate the exercise again, but using F, G and H for using the result again in b).
Let

$$0 \to F \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 0 \tag{3}$$

be an exact sequence of R-Modules and N a free R-Module.
We want to show that the sequence

$$0 \to F \otimes_R N \xrightarrow{\alpha \otimes id_N} G \otimes_R N \xrightarrow{\beta \otimes id_N} H \otimes_R N \to 0 \tag{4}$$

is exact.
We want to show the following things:

i) $\alpha \otimes id_N$ is injective
ii) $\beta \otimes id_N$ is surjective
iii) $\mathrm{Im}(\alpha \otimes id_N) = \mathrm{Ker}(\beta \otimes id_N)$

proof of i). Since N is free it has a basis. Let $\mathcal{N} := \{ n_1 \mid i \in \mathcal{I}\}$ be this (potentially infinite) basis. Then we have an isomorphism $F \otimes N \cong \bigoplus_{n \in \mathcal{N}} F$ and we can write each element in $F \otimes N$ as a unique linear combination of $f_i \otimes n_i$ for $n_i \in \mathcal{N}$.
So let $\sum_{i \in \mathcal{I}} f_i \otimes n_i \in \mathrm{Ker}(\alpha \otimes id_N)$ be arbitrary. Since $\mathcal{N}$ is a basis only finitely many f_i are non zero and we can thus write $\sum_{i \in \mathcal{J}} f_i \otimes n_i$ for some finite index set $\mathcal{J}$. Since it is in the kernel we have $\alpha \otimes id_N(\sum_{i \in \mathcal{J}} f_i \otimes n_i) = \sum_{i \in \mathcal{J}} \alpha(f_i) \otimes n_i = 0$. We note that elements of the form $\alpha(f_i) \otimes n_i$ in $G \otimes N \cong \bigoplus_{n \in \mathcal{N}} G$ also build unique linear combinations of elements and thus $\sum_{i \in \mathcal{J}} \alpha(f_i) \otimes n_i = 0 \iff \alpha(f_i) \otimes n_i = 0$ $\forall i$. This means there are index sets $\mathcal{A}, \mathcal{B}$ with $\mathcal{A} \cup \mathcal{B} = \mathcal{J}$ such that $\alpha(f_k) = 0$ for all $k \in \mathcal{A}$ and $id_N(n_j) = 0$ for all $j \in \mathcal{B}$. Since α is injective $\alpha(f_k) = 0 \iff f_k = 0$ and $id_N(n_j) = 0 \iff n_j = 0$ [1]. Thus the preimage $\sum_{i \in \mathcal{J}} f_i \otimes n_i = 0$. Because we took an arbitrary element in the kernel and showed that it is equal to zero we conclude that $\alpha \otimes id_N$ is injective.

□

proof of ii) and iii). This follows from Problem 2a). □

4b). We note that $C_i(X)$ is a free $\mathbb{Z}$-Module and each abelian group is a $\mathbb{Z}$-Module. By using the fact that every $C_i(X)$ is free abelian and changing N to $C_i(X)$ in equation (4) in a) for every i we obtain a SES of chain complexes

$$0 \to F \otimes_{\mathbb{Z}} C_*(X) \xrightarrow{\alpha \otimes id_{C_*(X)}} G \otimes_{\mathbb{Z}} C_*(X) \xrightarrow{\beta \otimes id_{C_*(X)}} H \otimes_{\mathbb{Z}} C_*(X) \to 0. \tag{5}$$

Using the LES-Theorem we obtain a LES

$$\cdots \to H_n(X; F) \xrightarrow{\alpha_*} H_n(X; G) \xrightarrow{\beta_*} H_n(X; H) \xrightarrow{\partial_*} \ldots \tag{6}$$

[1] Since n_j are basis elements $n_j \neq 0$ for all $j \in \mathcal{I}$ we are never in this case, but it was added for the sake of completeness.

4c). We remember that the sequence

$$0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}/n\mathbb{Z} \to 0 \tag{7}$$

is exact, with f being the multiplication with n and g given by $1 \mapsto [1]$.
Now plugging this into the LES (6) we get from part b) the following LES:

$$\ldots \to H_i(X;\mathbb{Z}) \xrightarrow{f_{*,i}} H_i(X;\mathbb{Z}) \xrightarrow{g_*} H_i(X;\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\partial} H_{i-1}(X;\mathbb{Z}) \xrightarrow{f_{*,i-1}} H_{i-1}(X;\mathbb{Z}) \to \ldots \tag{8}$$

From any LES we can extract a SES in the following way:
Let

$$\ldots \to A \xrightarrow{\beta} B \xrightarrow{\gamma} C \xrightarrow{\delta} D \to \ldots \tag{9}$$

be a LES.
Then we get

$$0 \to B/\mathrm{Im}(\beta) \xrightarrow{\tilde{\gamma}} C \xrightarrow{\tilde{\delta}} \mathrm{Im}(\delta) \to 0. \tag{10}$$

Since $\mathrm{Im}(\beta) = \mathrm{Ker}(\gamma)$, the induced map $\tilde{\gamma}$ is an isomorphism between $B/\mathrm{Im}(\beta)$ and $\mathrm{Im}(\gamma)$ and therefore injective. Clearly δ maps surjectively onto its image, therefore $\tilde{\delta}$ the map induced by δ is surjective. The exactness at C follows from the exactness at C in the LES (9).
Now realize that $f_{*,i-1}$ is also the multiplication with n and therefore $\mathrm{Ker}(f_{*,i-1})$ is exactly $\mathrm{Tors}_n(H_{i-1}(X;\mathbb{Z}))$. Due to exactness we have:

$$\mathrm{Im}(\partial) = \mathrm{Ker}(f_{*,i-1}) = \mathrm{Tors}_n(H_{i-1}(X;\mathbb{Z})). \tag{11}$$

Using $\mathrm{Im}(f_{*,i}) = nH_i(X;\mathbb{Z})$ and (11) we can extract the following SES from (8):

$$0 \to H_i(X;\mathbb{Z})/nH_i(X;\mathbb{Z}) \to H_i(X;\mathbb{Z}/n\mathbb{Z}) \to \mathrm{Tors}_n(H_{i-1}(X;\mathbb{Z})) \to 0. \tag{12}$$

Problem 5

no solutions for starred problems