Problem 1

by Vladimir Nowak

I want to thank Semyon for his insightful comments about my solution attempts to the exercises. Throughout the following, we refer to the map $h\colon S^{2n+1} \to \mathbf{C}P^n$ as the Hopf-fibration.

a).

Proof. Define the map:

$$\varphi\colon D^{2n+2} \subset \mathbf{C}^{n+1} \to \mathbf{C}P^{n+1}, (z_0, \ldots, z_n) \mapsto \left[z_0 \colon \cdots \colon z_n \colon 1 - \sum_{j=0}^{n} |z_j|^2 \right].$$

This is certainly a continuous map, given as the composition of maps

$$D^{2n+2} \hookrightarrow \mathbf{C}^{n+1} \to \mathbf{C}^{n+2} \xrightarrow{\pi} \mathbf{C}P^{n+1},$$

with π the canonical projection map. We notice that on $S^{2n+1} \subset D^{2n+2}$, φ yields the Hopf-fibration, i.e. $\varphi(S^{2n+1}) = \mathbf{C}P^n \subset \mathbf{C}P^{n+1}$, meaning that φ induces a map $\hat{\varphi}\colon \mathbf{C}P^n \cup_h D^{2n+2} \to \mathbf{C}P^{n+1}$. Therefore, in order to show that $\mathbf{C}P^{n+1} \cong \mathbf{C}P^n \cup_h D^{2n+2}$, it suffices to show that the induced map $\hat{\varphi}$ is a homeomorphism. We remark that since the domain of the map $\hat{\varphi}$ is compact (as the quotient of a compact space $\mathbf{C}P^n \sqcup D^{2n+2}$) and the codomain is Hausdorff, it is enough to show that $\hat{\varphi}$ is bijective. More specifically, it suffices to show that $\varphi|_{\mathring{D}^{2n+2}}$ bijects onto $\mathbf{C}P^{n+1} - \mathbf{C}P^n$, seeing as $\hat{\varphi}|_{\mathbf{C}P^n} \colon \mathbf{C}P^n \to \hat{\varphi}(\mathbf{C}P^n)$ is bijective.
We first check surjectivity. Let $\left[z'_0 \colon \cdots \colon z'_n \colon z'_{n+1}\right] \in \mathbf{C}P^{n+1} - \mathbf{C}P^n$, i.e. the last entry fulfils $z'_{n+1} \neq 0$. Let $r := \sqrt{\sum_{j=0}^{n+1} \left|z'_j\right|^2}$ and $e^{i\alpha} \in S^1$ be the phase, such that $\frac{e^{i\alpha} z'_{n+1}}{r} \in \mathbf{R}_{>0}$. We then rescale the representative of the class $\left[z'_0 \colon \cdots \colon z'_n \colon z'_{n+1}\right]$ by $\frac{e^{i\alpha}}{r}$ and show that $(z_0 \ldots, z_n) \in \mathbf{C}^{n+1}$ defined through the system:

$$\begin{cases} 1 - \sum_{j=0}^{n} |z_j|^2 = \frac{e^{i\alpha}}{r} z'_{n+1} \\ z_j = \frac{e^{i\alpha}}{r} z'_j,\ 0 \leq j \leq n \end{cases} ;$$

in fact, has a solution in the interior of D^{2n+2}. Through this system of equations, we get a quadratic equation in the "variable" r of the form:

$$\begin{aligned} 0 &= r^2 - \left(e^{i\alpha} z'_{n+1}\right) r - \sum_{j=0}^{n} \left|z'_j\right|^2 \\ &= r^2 - \beta r - \sum_{j=0}^{n} \left|z'_j\right|^2 . \end{aligned}$$

We then get that the only legal solution for r (since it has to be positive) is $r = \frac{\beta + \sqrt{\beta^2 + 4\sum_{j=0}^{n} |z'_j|^2}}{2}$ and plugging this into the sum of squares, we get with $\beta > 0$:

$$\sum_{j=0}^{n} |z_j|^2 = \frac{4\sum_{j=0}^{n} |z'_j|}{\left(\beta + \sqrt{\beta^2 + 4\sum_{j=0}^{n} |z'_j|^2}\right)^2} < 1.$$

We conclude that φ maps the interior of D^{2n+2} surjectively onto $\mathbf{C}P^{n+1} - \mathbf{C}P^n$. Now we turn to the injectivity of φ restricted to $\mathring{D}^{2n+2}$, where the calculation is of a similar nature to the one performed for the surjectivity. Let $(z_0, \ldots, z_n), (z'_0 \ldots, z'_n) \in \mathring{D}^{2n+2}$ such that:

$$\left[z_0 \colon \cdots \colon z_n \colon 1 - \sum_{j=0}^{n} |z_j|^2\right] = \left[z'_0 \colon \cdots \colon z'_n \colon 1 - \sum_{j=0}^{n} |z'_j|^2\right].$$

By the definition of the complex projective space, there exists a $re^{i\alpha} \in \mathbf{C} - 0$ such that:

$$\begin{cases} re^{i\alpha}\left(1 - \sum_{j=0}^{n} |z_j|^2\right) = 1 - \sum_{j=0}^{n} |z'_j|^2 \\ re^{i\alpha} z_j = z'_j,\ 0 \le j \le n \end{cases};$$

From the first part of the system of equations we can deduce that $e^{i\alpha} = 1$. We thus end up with a quadratic equation in r of the form:

$$r^2 - r + \sum_{j=0}^{n} |z'_j|^2 (r - 1) = 0.$$

The only positive solution is $r = 1$ and we get the equality of points $(z_0, \ldots, z_n) = (z'_0 \ldots, z'_n)$, i.e. we also get injectivity. This concludes the proof. □

b).

Proof. From the previous exercise, we know that (for $n \ge 1$):

$$\mathbf{C}P^n \cong \mathbf{C}P^{n-1} \cup_h D^{2n} \cong \cdots \cong \left(\cdots\left(\mathbf{C}P^0 \cup_h D^2\right) \cup_h D^4\right) \cup_h D^6 \cdots\right) \cup_h D^{2n}.$$

We remark that $\mathbf{C}P^0 = \{*\}$ is just the pointed space, meaning the homology becomes:

$$H_k\left(\mathbf{C}P^0; M\right) = \begin{cases} M & k = 0 \\ 0 & \text{o/w} \end{cases}.$$

From the above construction, we see that a CW-structure on $\mathbf{C}P^n$ is given through $n+1$ cells, one 0-cell $\mathbf{C}P^0$ and n 2k-cells D^{2k} for $1 \le k \le n$. Furthermore, by Theorem 2.13 from lecture, we have $H_\bullet^{CW}(\mathbf{C}P^n; M) \coloneqq H_\bullet\left(C^{CW}(\mathbf{C}P^n) \otimes M\right) \cong H_\bullet(\mathbf{C}P^n; M)$, meaning going through cellular homology gives us the same homology w.r.t. coefficients M. We get the chain complex:

$$0 \to C_{2n}^{CW}(\mathbf{C}P^n) \otimes M \xrightarrow{d} C_{2n-1}^{CW}(\mathbf{C}P^n) \otimes M \xrightarrow{d} \cdots C_1^{CW}(\mathbf{C}P^n) \otimes M \xrightarrow{d} C_0^{CW}(\mathbf{C}P^n) \otimes M \to 0.$$

As there are no cells in uneven dimensions, all uneven dimensions are trivial and using that $\mathbf{Z}\otimes M\cong M$ we get:

$$H_k(\mathbf{C}P^n;M)=\begin{cases}M & k \text{ even, and } k\leq 2n\\ 0 & \text{o/w}\end{cases}.$$

□

c).

Proof. We remark that the Hopf-fibration h is certainly a continuous and surjective map. As the open set for $[z_0\colon \cdots\colon z_n]\in\mathbf{C}P^n$, take $U_i:=\{[z_0\colon \cdots\colon z_n]\in\mathbf{C}P^n\colon z_i\neq 0\}$, wherever the i-th entry is non-zero. Note that

$$h^{-1}(U_i)=S^{2n+1}\cap\left\{(z_0,\dots,z_n)\in\mathbf{C}^{n+1}-0\colon z_i\neq 0\right\}$$

consists of all points on S^{2n+1} s.t. $z_i\neq 0$. Define a map:

$$\varphi_i\colon h^{-1}(U_i)\to U_i\times S^1,(z_0,\dots,z_n)\mapsto\left([z_0\colon \cdots\colon z_n],\frac{z_i}{|z_i|}\right).$$

Its continuous inverse is given through:

$$\psi_i\colon U_i\times S^1\to h^{-1}(U_i),\left([z_0\colon \cdots z_{i-1}\colon 1\colon z_{i+1}\colon \cdots\colon z_n],e^{it}\right)\mapsto\frac{e^{it}(z_0,\dots,1,\dots,z_n)}{\sqrt{1+\sum_{j=0,j\neq i}^n|z_j|^2}}.$$

This concludes the proof. □

Problem 2

by Sina Keller and Tristan Lovsin

2a). We know that $\mathbb{R}P^n\cong D^n/\sim$ with $\sim$ being the equivalence relation between antipodal points on ∂D^n. Denote $p\colon S^n\to D^n$ the projection map. [1] Now we define $h_{\mathbb{R}}$:

$$\begin{aligned}h_{\mathbb{R}}\colon S^n&\longrightarrow\mathbb{R}P^n\cong D^n/\sim\\ x&\longmapsto\begin{cases}[x]_\sim & x\text{ on the equator of }S^n\\ p(x) & x\text{ in the left hemisphere}\\ p(-x) & x\text{ in the right hemisphere}\end{cases}\end{aligned}$$

In order to show that $h_{\mathbb{R}}$ is a covering, we need to show that there exists a discrete fibre $F:=S^0=\{-1,1\}$, such that for any $x\in\mathbb{R}P^n$ there exists a neighbourhood $\tilde{U}$ of x and a homeomorphism φ such that the following diagram commutes:

[1] The projection is defined in the following way. We look at $S^n:=\{x\in\mathbb{R}^{n+1},\|x\|=1\}$ and $D^n:=\{x\in\mathbb{R}^{n+1},\|x\|\leq 1\}$ as subspaces of $\mathbb{R}^{n+1}$. Then $p(x_0,\dots,x_n)=(0,x_1,\dots,x_n)$.

$$\begin{array}{ccc} h_{\mathbb{R}}^{-1}(\tilde{U}) & \xrightarrow{\varphi} & \tilde{U} \times S^0 \\ & {\scriptstyle h_{\mathbb{R}}} \searrow \quad \swarrow {\scriptstyle P_r} & \\ & \tilde{U} & \end{array}$$

In order to determine a useful neighborhood, we will distinguish two cases: one where $h_{\mathbb{R}}^{-1}(x)$ is on the equator of S^n and the other where that is not the case. If $h_{\mathbb{R}}^{-1}(x)$ is not on the equator, then let $\tilde{U}_x := h_{\mathbb{R}}(S^n \setminus \{(x_0, \ldots, x_n) \in S^n \mid x_0 = 0\})$ and if $h_{\mathbb{R}}^{-1}(x)$ is on the equator, then let

$$\tilde{U}_x := h_{\mathbb{R}}(S^n \setminus \{(x_0, \ldots, x_n) \in S^n \mid \langle y, (x_0, \ldots, x_n)\rangle = 0 \ \forall y \in h_{\mathbb{R}}^{-1}(x)\}),$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^n$.
Now let $x \in \mathbb{R}P^n$ and $\tilde{U}_x \subset \mathbb{R}P^n$ the open neighborhood we just defined. If $h_{\mathbb{R}}^{-1}(x)$ is not on the equator we denote U_x the right hemisphere and U_{-x} the left hemisphere and if $h_{\mathbb{R}}^{-1}(x)$ is on the equator, then let $U_x := \{(x_0, \ldots, x_n) \in S^n \mid \langle x, (x_0, 0, \ldots, 0)\rangle \geq 0\}$ and $U_{-x} := \{(x_0, \ldots, x_n) \in S^n \mid \langle -x, (x_0, 0, \ldots, 0)\rangle \geq 0\}$. Then $h_{\mathbb{R}}^{-1}(\tilde{U}_x) = U_x \bigsqcup U_{-x}$ for both versions of $\tilde{U}_x$. We define φ for both versions as

$$\begin{aligned} \varphi \colon U_x \bigsqcup U_{-x} &\longrightarrow \tilde{U}_x \times S^0 \\ z &\longmapsto \begin{cases} ([z], 1) & \iff z \in U_x \\ ([z], -1) & \iff z \in U_{-x} \end{cases} \end{aligned}$$

and ψ for both versions as

$$\begin{aligned} \psi \colon \tilde{U}_x \times S^0 &\longrightarrow U_x \bigsqcup U_{-x} \\ ([z], y) &\longmapsto yz \end{aligned}$$

Let $y \in U_x$ and $-y \in U_{-x}$, then

$$\left.\begin{aligned} \psi(\varphi(y)) = \psi([y], 1) = y \\ \psi(\varphi(-y)) = \psi([y], -1) = -y \end{aligned}\right\} = \mathrm{id}_{h_{\mathbb{R}}^{-1}(\tilde{U}_x)} \text{ and}$$

$$\left.\begin{aligned} \varphi(\psi([y], 1)) = ([y], 1) \\ \varphi(\psi([y], -1)) = \varphi(-y) = ([y], -1) \end{aligned}\right\} = \mathrm{id}_{\tilde{U}_x \times S^0}$$

Therefore we have found an homeomorphism between $\tilde{U} \times S^0$ and $h_{\mathbb{R}}^{-1}(\tilde{U})$ and have a covering as desired.

2b). We choose the disc presentation of $\mathbb{R}P^{n+1} \cong D^{n+1}/\sim$ with $x \sim y$ iff $x = y$ or $x = -y \ \forall x, y \in \partial D^{n+1}$. In $\mathbb{R}P^n \cup_{h_{\mathbb{R}}} D^{n+1}$ we have that $x \sim h_{\mathbb{R}}(x) = [x] = h_{\mathbb{R}}(-x) \sim x$ and thus $x \sim y$ iff $x = y$ or $x = -y \ \forall x, y \in \partial D^{n+1}$, obtaining $\mathbb{R}P^n \cup_{h_{\mathbb{R}}} D^{n+1} \cong D^{n+1}/\sim \ \cong \mathbb{R}P^{n+1}$ as desired.

2c). ${\mathbb{R}P^n}/{\mathbb{R}P^{n-1}} \cong {(\mathbb{R}P^{n-1} \cup_{h_\mathbb{R}} D^n)}/{\mathbb{R}P^{n-1}} \cong {D^n}/{\backsim} \cong S^n$ where $x \backsim y$ iff $x, y \in \partial D^n$.
Now we compute the degree of the map $\pi \circ h_\mathbb{R}$ with π denoting the projection from $\mathbb{R}P^n$ to $\mathbb{R}P^n/\mathbb{R}P^{n-1}$. Since $\pi \circ h_\mathbb{R}$ is a smooth map from S^n to S^n we know from Algebraic Topology I that

$$\deg(\pi \circ h_\mathbb{R}) = \sum_{j=1}^{k} \mathcal{E}_{q_j}(\pi \circ h_\mathbb{R})$$

where $\mathcal{E}_{q_j}(\pi \circ h_\mathbb{R})$ is the local degree of $\pi \circ h_\mathbb{R}$ at q_j with $q_j \in \pi \circ h_\mathbb{R}^{-1}(p)$ for some regular value $p \in S^n$.
Let $p \in S^n$ be arbitrary and fixed. Then $\pi \circ h_\mathbb{R}^{-1}(p) = \{-p, p\}$. Clearly $\mathcal{E}_p(\pi \circ h_\mathbb{R}) = 1$, as $\pi \circ h_\mathbb{R}$ acts like the identity in a small neighbourhood of p. Furthermore, $\mathcal{E}_{-p}(\pi \circ h_\mathbb{R}) = (-1)^{n+1}$ as $\pi \circ h_\mathbb{R}$ acts as the antipodal map in a small neighbourhood of $-p$. Thus we have

$$\deg(\pi \circ h_\mathbb{R}) = 1 + (-1)^{n+1} = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}$$

2d). We choose the standard CW-complex on $\mathbb{R}P^n$ with 1 p-cell in each dimension $p \leq n$ and 0 otherwise.
For $M = \mathbb{Z}$ we obtain

$$\dots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \dots$$

if n is even and

$$\dots \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \dots$$

if n is odd. This yields

$$H_p(\mathbb{R}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } p = n \text{ where } n \text{ is odd or } p = 0 \\ \mathbb{Z}_2 & \text{if } 0 < p \leq n-1 \text{ and } p \text{ is odd} \\ 0 & \text{if } p > n \text{ or } 0 < p \leq n \text{ and } p \text{ is even} \end{cases}$$

For $M = \mathbb{Z}_2$ we have

$$\dots \to 0 \to \mathbb{Z}_2 \xrightarrow{2} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{2} \dots$$

if n is even and

$$\dots \to 0 \to \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{2} \mathbb{Z}_2 \xrightarrow{0} \dots$$

if n is odd. However $2 \equiv 0$ in $\mathbb{Z}_2$ and thus we have

$$\dots \to 0 \to \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \dots$$

regardless of n. Hence

$$H_p(\mathbb{R}P^n; \mathbb{Z}_2) = {\mathrm{Ker}(d_p)}/{\mathrm{Im}(d_{p+1})} \cong \begin{cases} {\mathbb{Z}_2}/{0} = \mathbb{Z}_2 & \text{if } p \leq n \\ 0 & \text{if } p > n \end{cases}$$

PROBLEM 3

by Maria Morariu

a). We prove the statement using the naturality of the Gysin long exact sequence. Take an arbitrary map $f\colon \mathbb{R}P^n \to \mathbb{R}P^m$ and let $p_n\colon S^n \to \mathbb{R}P^n$, $p_m\colon S^m \to \mathbb{R}P^m$ be the usual two-coverings. We have a map $f \circ p_n\colon S^n \to \mathbb{R}P^m$ and the fundamental group of S^n is trivial ($n > 1$), so by the lifting property of covers, there exists a function $g\colon S^n \to S^m$ such that $p_m \circ g = f \circ p_n$.
The Gysin LES gives the following commutative diagram, with exact rows:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_p(\mathbb{R}P^n;\mathbb{Z}/2) & \xrightarrow{T_*} & H_p(S^n;\mathbb{Z}/2) & \xrightarrow{p_{n,*}} & H_p(\mathbb{R}P^n;\mathbb{Z}/2) & \xrightarrow{\partial} & H_{p-1}(\mathbb{R}P^n;\mathbb{Z}/2) & \longrightarrow \cdots \\
& \downarrow f_* & & \downarrow g_* & & \downarrow f_* & & & \\
\cdots \longrightarrow & H_p(\mathbb{R}P^m;\mathbb{Z}/2) & \xrightarrow{T'_*} & H_p(S^m;\mathbb{Z}/2) & \xrightarrow{p_{m,*}} & H_p(\mathbb{R}P^m;\mathbb{Z}/2) & \xrightarrow{\partial'} & H_{p-1}(\mathbb{R}P^m;\mathbb{Z}/2) & \longrightarrow \cdots
\end{array}$$

As in the lecture, we get the following diagrams with exact rows:
For $2 \leq p \leq m-1$ (1):

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_p(\mathbb{R}P^n;\mathbb{Z}/2) & \longrightarrow & H_{p-1}(\mathbb{R}P^n;\mathbb{Z}/2) & \longrightarrow & 0 \\
& & \downarrow f_{*,p} & & \downarrow f_{*,p-1} & & \\
0 & \longrightarrow & H_p(\mathbb{R}P^m;\mathbb{Z}/2) & \longrightarrow & H_{p-1}(\mathbb{R}P^m;\mathbb{Z}/2) & \longrightarrow & 0
\end{array}$$

and (2):

$$\begin{array}{ccccccccc}
0 \longrightarrow & H_m(\mathbb{R}P^n;\mathbb{Z}/2) & \xrightarrow{T_*} & H_m(S^n;\mathbb{Z}/2) & \xrightarrow{p_{n,*}} & H_m(\mathbb{R}P^n;\mathbb{Z}/2) & \xrightarrow{\partial} & H_{m-1}(\mathbb{R}P^n;\mathbb{Z}/2) & \longrightarrow 0 \\
& \downarrow f_{*,m} & & \downarrow g_* & & \downarrow f_{*,m} & & \downarrow f_{*,m-1} & \\
0 \longrightarrow & H_m(\mathbb{R}P^m;\mathbb{Z}/2) & \xrightarrow{T'_*} & H_m(S^m;\mathbb{Z}/2) & \xrightarrow{p_{m,*}} & H_m(\mathbb{R}P^m;\mathbb{Z}/2) & \xrightarrow{\partial'} & H_{m-1}(\mathbb{R}P^m;\mathbb{Z}/2) & \longrightarrow 0
\end{array}$$

Now m and n are different, so $H_m(S^n) = 0$. Thus, commutativity of the first square in (2) implies $T'_* \circ f_{*,m} = 0$. By the lecture, $H_p(\mathbb{R}P^n;\mathbb{Z}/2) \cong H_p(\mathbb{R}P^m;\mathbb{Z}/2) \cong \mathbb{Z}/2$ for all $p \leq m$. By exactness T'_* is injective, so we have $f_{*,m} = 0$.
Now ∂ and ∂' are surjective homomorphisms $\mathbb{Z}/2 \to \mathbb{Z}/2$, so they are isomorphisms. Thus commutativity of the last square in (2), together with $f_{*,m} = 0$ give $f_{*,m-1} = 0$. Now exactness of the rows in diagram (1) shows that the horizontal maps are isomorphisms. Inductively, this gives $f_{*,p} = 0$ for all $1 \leq p \leq m$. In particular, $f_{*,1} = 0$, which is what we wanted to show.

b). Assume that $\mathbb{R}P^m$ was a retract of $\mathbb{R}P^n$, i.e. there exists a continuous map $r\colon \mathbb{R}P^n \to \mathbb{R}P^m$, such that the canonical inclusion $i\colon \mathbb{R}P^m \to \mathbb{R}P^n$ satisfies $r \circ i = id_{\mathbb{R}P^m}$.
By functoriality, this implies $r_* \circ i_* = id_{H_p(\mathbb{R}P^m;\mathbb{Z}/2)}, \forall p \geq 2$. In particular, $r_*\colon H_1(\mathbb{R}P^n;\mathbb{Z}/2) \to H_1(\mathbb{R}P^m;\mathbb{Z}/2)$ is a surjection. However, a) implies that this map is zero. By the lecture, the homology groups are non-zero, so this is a contradiction. Therefore $\mathbb{R}P^m$ cannot be a retract of $\mathbb{R}P^n$.

Exercise 4. (Sheet 2)

(a) Define $f : \mathbb{RP}^2 \longrightarrow S^2$ to be the quotient map

$$\mathbb{RP}^2 \longrightarrow \mathbb{RP}^2/\mathbb{RP}^1 \cong S^2.$$

$\Longrightarrow$ The interior of the 2-cell maps homeomorphically onto the 2-cell.

$\Longrightarrow$ The induced map on cell complexes

$$\begin{array}{rcccccl} & & 2 & & 1 & & 0 \\ C_\bullet^{cw}(\mathbb{RP}^2;\mathbb{F}_2): & & \mathbb{F}_2 & \longrightarrow & \mathbb{F}_2 & \longrightarrow & \mathbb{F}_2 \\ & & \downarrow \mathbb{1} & & \downarrow & & \downarrow \mathbb{1} \\ C_\bullet^{cw}(S^2;\mathbb{F}_2): & & \mathbb{F}_2 & \longrightarrow & 0 & \longrightarrow & \mathbb{F}_2 \end{array} \quad \Longrightarrow \quad f_* : H_2(\mathbb{RP}^2;\mathbb{F}_2) \xrightarrow{\cong} H_2(S^2;\mathbb{F}_2).$$

(b) Consider $f : \mathbb{RP}^2 \longrightarrow S^2$

constant map: $\mathbb{RP}^2 \longrightarrow \{n\} \in S^2$

the induced maps

$$H_2(\mathbb{RP}^2;\mathbb{F}_2) \underset{0}{\overset{f_* = id}{\rightrightarrows}} H_2(S^2;\mathbb{F}_2)$$

on the other hand

$$\begin{array}{ccc} H_2(\mathbb{RP}^2;\mathbb{Z}) & \longrightarrow & H_2(S^2;\mathbb{Z}) \cong \mathbb{Z} \\ \| & \uparrow & \\ 0 & \text{always trivial} & \\ & \downarrow & \\ H_1(\mathbb{RP}^2;\mathbb{Z}) & \longrightarrow & H_1(S^2;\mathbb{Z}) \cong 0 \\ H_0(\mathbb{RP}^2;\mathbb{Z}) & \longrightarrow & H_0(S^2;\mathbb{Z}) \\ \wr\| & \uparrow & \wr\| \\ \mathbb{Z} & \text{always the same isomorphism} & \mathbb{Z} \end{array}$$

PROBLEM 5

by Noah Stäuble & Philip Sandt

A Borsuk-Ulam type statement does not hold for the Torus $S_1 \times S_1$ - there exists a continuous function $S^1 \times S^1 \to \mathbb{R}^2$ where no antipodal points have the same image. To see this, consider the following counter example

$$\begin{aligned} \Phi : S_1 \times S_1 \to S_1 \hookrightarrow \mathbb{R}^2 \\ (s,t) \mapsto s \mapsto s \end{aligned} \tag{1}$$

Φ is continuous and satisfies the following property: if (s,t) and $(-s,-t)$ have the same image, then this image is s and $-s$ at the same time, so it is zero, hence

$$\Phi(s,t) = s = 0 \in \mathbb{R}^2.$$

But 0 is not in S^1 so we cannot have antipodal points mapping to the same point.

Problem 6

by Naomi Rosenberg

Fix $n \in \mathbb{N}$. Let $\bigcup_{k=1}^{n+1} U_k$ be a covering of S^n with closed sets U_k. Without loss of generality, assume that U_k does not contain any antipodal points for $k \in \{1, ..., n\}$; otherwise the claim from the exercise is already satisfied.

Our goal is to construct a map $f : S^n \to \mathbb{R}^n$, to then apply the Borsuk-Ulam Theorem in order to deduce that there exists a point $x \in S^n$ satisfying $f(x) = f(-x)$, and to then notice that both, x and $-x$, have to be contained in U_{k+1}.

We start by defining

$$f_k : S^n \to \mathbb{R},\ f_k(x) \coloneqq d(x, U_k) \coloneqq \inf_{y \in U_k} d(x, y),$$

for every $k \in \{1, ..., n\}$, where we denote by d the Euclidean distance on $\mathbb{R}^n$. We claim that f_k is continuous. Indeed, d is continuous and for every $x, y \in S^n$, the following holds:

$$\begin{aligned} f_k(x) = d(x, U_k) &= \inf_{u \in U_k} (d(x, u)) \\ &\leq \inf_{u \in U_k} (d(x, y) + d(y, u)) = d(x, y) + d(y, U_k) = d(x, y) + f_k(y), \end{aligned}$$

and analogously $f_k(y) \leq d(x, y) + f_k(x)$. Thus, $|f_k(x) - f_k(y)| \leq d(x, y)$. Hence, f_k is 1-Lipschitz and therefore continuous.
Next, we define the function f as follows:

$$f : S^n \to \mathbb{R}^n,\ f(x) \coloneqq (f_1(x), ..., f_n(x)).$$

Since f is continuous in every component, it is continuous on its whole domain. Thus, we can conclude with the Borsuk-Ulam Theorem that there exists an element $x \in S^n$ satisfying $f(x) = f(-x)$. In particular, by the definition of f, x satisfies $d(x, U_k) = f_k(x) = f_k(-x) = d(-x, U_k)$ for every $k \in \{1, ..., n\}$. By assumption, U_k does not contain any antipodal points, thus U_k cannot contain both, x and $-x$. In fact, neither x, nor $-x$ can be contained in U_k. Indeed, if $x \in U_k$, then $0 = d(x, U_k) = d(-x, U_k)$ and therefore $-x \in U_k$, which is a contradiction.
Since the above holds for every $k \in \{1, ...n, \}$, x and $-x$ are contained in none of the U_k's for $k \in \{1, ..., n\}$.
Taking into account that $\bigcup_{k=1}^{n+1} U_k$ is a covering of S^n, the above implies that $x, -x \in U_{k+1}$. This concludes the proof since consequently, x and y are in the same set from the covering.