# DR. LUKAS LEWARK ALGEBRAIC TOPOLOGY II SOLUTIONS SHEET 3 ETH ZÜRICH SPRING, 2024

#### Problem 1

Leon Dahlmeier

We want to show  $\operatorname{Tor}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/g\mathbb{Z}$  for  $g \coloneqq \operatorname{gcd}(m, n)$ .  $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{d_0} \mathbb{Z}/n\mathbb{Z} \to 0$  is a free resolution of  $\mathbb{Z}/n\mathbb{Z}$  called F. Where  $\xrightarrow{n}$  is multiplication with n and  $d_0$  the projection. Tensoring the deleted resolution  $F^{\mathbb{Z}/n\mathbb{Z}}$ with  $\mathbb{Z}/m\mathbb{Z}$  yields:

$$0 \to \mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} \xrightarrow{n \otimes id_{\mathbb{Z}/m\mathbb{Z}}} \mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} \to 0.$$

After simplifying everything we already know about the Tensor product:

 $0 \xrightarrow{0} \mathbb{Z}/m\mathbb{Z} \xrightarrow{n} \mathbb{Z}/m\mathbb{Z} \to 0$ 

Therefore  $\operatorname{Tor}(\mathbb{Z}/m\mathbb{Z},\mathbb{Z}/n\mathbb{Z}) = H_1(F^{\mathbb{Z}/n\mathbb{Z}},\mathbb{Z}/m\mathbb{Z}) = \ker(\mathbb{Z}/m\mathbb{Z} \xrightarrow{n} \mathbb{Z}/m\mathbb{Z}).$ 

Finally, let us take a closer look at:  $\ker(\mathbb{Z}/m\mathbb{Z} \xrightarrow{n} \mathbb{Z}/m\mathbb{Z})$ . Remember  $g = \gcd(m, n)$  and let u and k be such that  $u \cdot g = m$  and  $k \cdot g = n$ . Since  $n \cdot u = k \cdot u \cdot g = k \cdot m \equiv 0 \pmod{m}$ , we have  $\operatorname{im}(\mathbb{Z}/m\mathbb{Z} \xrightarrow{n} \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/u\mathbb{Z}$ . We conclude using the isomorphism theorem:

$$\operatorname{Tor}(\mathbb{Z}/m\mathbb{Z},\mathbb{Z}/n\mathbb{Z}) \cong \frac{\mathbb{Z}/m\mathbb{Z}}{\mathbb{Z}/u\mathbb{Z}} \cong \mathbb{Z}/g\mathbb{Z}$$

### Problem 2

Leon Dahlmeier

**a).** Since  $\operatorname{Tor}(A, B) \cong \operatorname{Tor}(T(A), T(B))$  let us assume without loss of generality that A and B are torsion. Let us define  $C := \bigoplus_{0 \neq b \in B} \mathbb{Z}/\operatorname{ord}(b)\mathbb{Z}$  and  $f : C \to B$  by sending  $[1] \in \mathbb{Z}/\operatorname{ord}(b)\mathbb{Z}$  to b.

The abelian group A together with the short exact sequence:

$$0 \to \ker(f) \to C \xrightarrow{f} B \to 0$$

give rise to the following long exact sequence:

$$0 \to \operatorname{Tor}(A, \ker(f)) \xrightarrow{g} \operatorname{Tor}(A, C) \to \operatorname{Tor}(A, B) \to A \otimes \ker(f) \xrightarrow{h} A \otimes C \to A \otimes B \to 0$$

from which we can extract the short exact sequence:

$$0 \to \operatorname{coker}(g) \xrightarrow{\alpha} \operatorname{Tor}(A, B) \xrightarrow{\beta} \ker(h) \to 0$$

Solutions Sheet 3

But by 4.14.

$$\operatorname{Tor}(A,C) = \bigoplus_{0 \neq b \in B} \operatorname{Tor}(A, \mathbb{Z}/\operatorname{ord}(b)\mathbb{Z}) = \bigoplus_{0 \neq b \in B} \ker(A \xrightarrow{\cdot \operatorname{ord}(b)} A)$$

which implies that Tor(A, C) as a subgroup of a torsion group is torsion. Also  $A \otimes \text{ker}(f)$  is torsion since we assumed A to be. Which then means that coker(g) and ker(h) are, meaning Tor(A, B) is:

For  $x \in \text{Tor}(A, B) \exists n \in \mathbb{N}$  s.t.  $n\beta(x) = \beta(xn) = 0 \Rightarrow xn \in \text{ker}(\beta) = \text{im}(\alpha)$ . Hence,  $\exists y \in \text{coker}(g)$  s.t.  $\alpha(y) = xn$  but  $\exists m \in \mathbb{N}$  s.t. ym = 0. Meaning  $(xn)m = \alpha(y)m = \alpha(ym) = \alpha(0) = 0$ , which concludes the proof.

**b).** The long exact sequence for  $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$  and the abelian group T(A) is:

$$0 \to \operatorname{Tor}(T(A), \mathbb{Z}) \to \operatorname{Tor}(T(A), \mathbb{Q}) \to \operatorname{Tor}(T(A), \mathbb{Q}/\mathbb{Z}) \to T(A) \otimes \mathbb{Z} \to T(A) \otimes \mathbb{Q} \to T(A) \otimes \mathbb{Q}/\mathbb{Z} \to 0.$$

 $\mathbb{Z}$  and  $\mathbb{Q}$  are torsion-free and  $\mathbb{Z} \otimes T(A)$  is isomorphic to T(A). Further for  $a \otimes q \in T(A) \otimes \mathbb{Q}$  we have:  $a \in T(A)$  meaning there is an  $n \in \mathbb{Z} \setminus 0$  s.t.  $a \cdot n = 0$  i.e.  $a \otimes q = na \otimes \frac{q}{n} = 0$  implying  $T(A) \otimes \mathbb{Q} \cong 0$ . Using these isomorphisms we can simplify to:

$$0 \to 0 \to 0 \to \operatorname{Tor}(T(A), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\phi} T(A) \to T(A) \otimes \mathbb{Q} \to T(A) \otimes \mathbb{Q}/\mathbb{Z} \to 0$$

Which means that  $\phi$  an isomorphism.

### Problem 3

Aparna Jeyakumar

**a**). (a) From the Universal Coefficient Theorem, we have the following commutative diagram

Since  $-\otimes 1_M$  and  $\operatorname{Tor}(-, M)$  are additive functors between the category of abelian groups, they take isomorphisms to isomorphisms. In particular,  $f_* \otimes 1_M$  and  $\operatorname{Tor}(f_*, 1_M)$  are isomorphisms. Now, using the five lemma, we get that the vertical map in the middle  $f_* : H_n(X; M) \to H_n(Y; M)$  is an isomorphism.

**b).** To show that  $f_*: H_n(X) \to H_n(Y)$  is an isomorphism, it is enough to show that  $H_n(Mc(f)) = 0$  for all *n* where Mc(f) is the mapping cone complex of the map f. This is due to the following result from Homological Algebra:

If  $f: X_{\bullet} \to Y_{\bullet}$  is a chain map of complexes then, the induced map on the homology,  $f_*: H_n(X_{\bullet}) \to H_n(Y_{\bullet})$  is an isomorphism iff  $H_n(Mc(f)) = 0$  for all n, where Mc(f) is the mapping cone complex of the map f.

We have that  $f_*: H_n(X; \mathbb{Q}) \to H_n(Y, \mathbb{Q})$  is an isomorphism which implies that  $H_n(Mc(f \otimes 1_{\mathbb{Q}})) = 0$  for all n. Since  $\mathbb{Q}$  is torsion-free,  $\operatorname{Tor}(H_n(Mc(f)), \mathbb{Q}) = 0$  and from the UCT for Mc(f), we get that

$$H_n(Mc(f)) \otimes \mathbb{Q} \cong H_n(Mc(f); \mathbb{Q}) \cong H_n(Mc(f \otimes 1_{\mathbb{Q}})) \cong 0$$

(The second isomorphism is due to the distributive property of the tensor product over direct sums).

Similarly, we have  $H_n(Mc(f \otimes 1_{\mathbb{Z}_p})) = 0$  for all p prime, for all n. Using the UCT again, we get that

$$(H_n(Mc(f)) \otimes \mathbb{Z}_p) \oplus \operatorname{Tor}(H_{n-1}(Mc(f)), \mathbb{Z}_p) \cong 0$$

which implies that both the terms are 0 and in particular,  $\operatorname{Tor}(H_n(Mc(f)), \mathbb{Z}_p) = 0$ for all p prime and for all n. Setting  $A = H_n(Mc(f))$ , it is now enough to show that the following claim is true.

Claim : If A is an abelian group such that  $A \otimes \mathbb{Q} = 0$  and  $\operatorname{Tor}(A, \mathbb{Z}_p) = 0$  for all p prime, then A = 0.

*Proof* : Suppose  $A \otimes \mathbb{Q} = 0$  and  $\text{Tor}(A, \mathbb{Z}_p) = 0$  for all p prime. Consider the short exact sequences

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}_p \longrightarrow 0$$
$$0 \longrightarrow \mathbb{Z} \longleftrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

Then, we get the following LESs,

 $0 \to \operatorname{Tor}(A, \mathbb{Z}) \to \operatorname{Tor}(A, \mathbb{Z}) \to \operatorname{Tor}(A, \mathbb{Z}_p) \to A \otimes \mathbb{Z} \xrightarrow{\cdot p} A \otimes \mathbb{Z} \to A \otimes \mathbb{Z}_p \to 0$  $0 \to \operatorname{Tor}(A, \mathbb{Z}) \to \operatorname{Tor}(A, \mathbb{Q}) \to \operatorname{Tor}(A, \mathbb{Q}/\mathbb{Z}) \to A \otimes \mathbb{Z} \to A \otimes \mathbb{Q} \to A \otimes Q/\mathbb{Z} \to 0$ The first LES reduces to

 $0 \longrightarrow A \xrightarrow{.p} A \longrightarrow A \otimes \mathbb{Z}_p \longrightarrow 0$ 

The injectivity of the map  $A \xrightarrow{p} A$  for all p implies that A is a torsion-free group. Then,  $\operatorname{Tor}(A, \mathbb{Q}/\mathbb{Z}) \cong 0$  and the second LES reduces to  $0 \to A \to 0$  and so A = 0.

Exercise 4. (Sheef 3)  
(\*) Claim For any group G, we have 
$$\operatorname{Tar}(G, \mathbb{Q}) = 0$$
  
 $\Gamma_{f} \otimes \mathbb{Q} \longrightarrow F_{e} \otimes \mathbb{Q}$  is a monomorphism.  $\square$   
 $= F_{e} \otimes \mathbb{Q} \longrightarrow F_{e} \otimes \mathbb{Q}$  is a monomorphism.  $\square$   
 $\Rightarrow By \sqcup CT, \to H_{n}(X;2) \otimes \mathbb{Q} \to H_{n}(X;\mathbb{Q}) \to \operatorname{Ter}(H_{n}(X;2),\mathbb{Q}) \to \mathbb{Q}$   
 $\Rightarrow H_{n}(X;2) \otimes \mathbb{Q} \cong H_{n}(X;\mathbb{Q}).$   
(6) Easily follows from UCT.  
 $\equiv \operatorname{K}_{n}(X;2) \cong \mathbb{Z}^{\otimes k} \otimes (\bigoplus_{\substack{q \in \mathbb{Z}/(q^{k_{1}})}) \otimes \mathbb{F}_{p} \cong$   
 $= H_{n}(X;2) \otimes \mathbb{F}_{p} = (\mathbb{Z}^{\otimes k} \otimes (\bigoplus_{\substack{q \in \mathbb{Z}/(q^{k_{1}})}) \otimes \mathbb{F}_{p})$   
 $\Rightarrow H_{n}(X;2) \otimes \mathbb{F}_{p} = (\mathbb{Z}^{\otimes k} \otimes (\bigoplus_{\substack{q \in \mathbb{Z}/(q^{k_{1}})}) \otimes \mathbb{F}_{p})$   
 $\Rightarrow H_{n}(X;2) \otimes \mathbb{F}_{p} = \mathbb{Q} \xrightarrow_{\substack{q \in \mathbb{Z}/(q^{k_{1}})} \otimes \mathbb{F}_{p}} =$   
 $\cong \mathbb{F}_{p}^{\otimes m} \oplus (\bigoplus_{\substack{q \in \mathbb{Z}/(q^{k_{1}})} \otimes \mathbb{F}_{p})) \cong_{m} \mathbb{F}_{p}^{\otimes m} \otimes (\bigoplus_{\substack{q \in \mathbb{Z}/(q^{k_{1}})} \otimes \mathbb{F}_{p})$   
 $\cong \mathbb{F}_{p}^{\otimes m} \oplus (\bigoplus_{\substack{q \in \mathbb{Z}/(q^{k_{1}})} \otimes \mathbb{F}_{p})) \cong_{m} \mathbb{F}_{p}^{\otimes m} \otimes (\bigoplus_{\substack{q \in \mathbb{Z}/(q^{k_{1}})} \otimes \mathbb{F}_{p}))$   
 $\stackrel{\text{furthered}}{=} \mathbb{F}_{p} \xrightarrow_{\substack{q \in \mathbb{Z}/(q^{k_{1}})} \otimes \mathbb{F}_{p}} = \mathbb{T}_{m}(\mathbb{Z}^{\otimes m} \oplus (\bigoplus_{\substack{q \in \mathbb{Z}/(q^{k_{1}})}); \mathbb{F}_{p}))$   
 $\stackrel{\text{furthered}}{=} \operatorname{Tar}(\mathbb{Z}^{\otimes m} \oplus (\bigoplus_{\substack{q \in \mathbb{Z}/(q^{k_{1}})}); \mathbb{F}_{p}))$   
 $\stackrel{\text{furthered}}{=} \mathbb{T}_{p}(\mathbb{Q}^{\otimes m} \oplus \mathbb{Q}^{\otimes m}$ 

## Problem 5

## Naomi Rosenberg

We start by constructing a free resolution of  $\mathbb{Z}/2$  as a  $\mathbb{Z}/4$ -module. To that extent, note that  $\mathbb{Z}/2$  can be interpreted as  $2 + 4\mathbb{Z}$ , which is a submodule of  $\mathbb{Z}/4$ . Consider the following sequence:

$$\ldots \longrightarrow \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\operatorname{proj}} \mathbb{Z}/2 \longrightarrow 0.$$

Note that  $\ker(\cdot 2) = 2 + 4\mathbb{Z} = \operatorname{im}(\cdot 2)$  and  $\ker(\operatorname{proj}) = 2 + 4\mathbb{Z} = \operatorname{im}(\cdot 2)$ . Consequently, the sequence defined above is a long exact sequence and therefore defines a free resolution F of the  $\mathbb{Z}/4$ -module  $\mathbb{Z}/2$ .

We thus get the following deleted free resolution:

 $F^{\mathbb{Z}/2} = \dots \longrightarrow \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \longrightarrow 0.$ 

This enables us to compute  $\operatorname{Tor}_{n}^{\mathbb{Z}/4}(\mathbb{Z}/2,\mathbb{Z}/2)$ . By plugging into the definition, we obtain:

$$\operatorname{Tor}_{n}^{\mathbb{Z}/4}(\mathbb{Z}/2,\mathbb{Z}/2) = \operatorname{H}_{n}(F^{\mathbb{Z}/2};\mathbb{Z}/2) = \operatorname{H}_{n}(F^{\mathbb{Z}/2}\otimes\mathbb{Z}/2).$$

So in order to determine  $\operatorname{Tor}_{n}^{\mathbb{Z}/4}(\mathbb{Z}/2,\mathbb{Z}/2)$ , it is sufficient to consider the long exact sequence

$$F^{\mathbb{Z}/2} \otimes \mathbb{Z}/2 = \ldots \longrightarrow \mathbb{Z}/4 \otimes \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \otimes \mathbb{Z}/2 \longrightarrow \ldots \longrightarrow \mathbb{Z}/4 \otimes \mathbb{Z}/2 \longrightarrow 0.$$

In the sequence above, the homomorphisms are given by  $(\cdot 2) \otimes \operatorname{id}_{\mathbb{Z}/2}$ . Notice that by Problem Sheet 1, Problem 1, it holds that  $\mathbb{Z}/4 \otimes \mathbb{Z}/2 \cong \mathbb{Z}/\operatorname{gcd}(2,4) \cong \mathbb{Z}/2$  and the homomorphism is precisely the zero map. Consequently,  $\operatorname{H}_n(F^{\mathbb{Z}/2} \otimes \mathbb{Z}/2) \cong \operatorname{ker}(\cdot 0) / \operatorname{im}(\cdot 0) \cong (\mathbb{Z}/2)/0 \cong \mathbb{Z}/2$  for all  $n \geq 0$ . By the above, this yields

By the above, this yields

$$\operatorname{Tor}_{n}^{\mathbb{Z}/4}(\mathbb{Z}/2,\mathbb{Z}/2)\cong\mathbb{Z}/2,$$

for all  $n \ge 0$ .

Now let's calculate  $\operatorname{Ext}_{\mathbb{Z}/4}^n(\mathbb{Z}/2,\mathbb{Z}/2)$ . By definition, it holds that

$$\operatorname{Ext}_{\mathbb{Z}/4}^{n}(\mathbb{Z}/2,\mathbb{Z}/2) = \operatorname{H}^{n}(\operatorname{Hom}(F^{\mathbb{Z}/2},\mathbb{Z}/2)),$$

where

$$\operatorname{Hom}(F^{\mathbb{Z}/2},\mathbb{Z}/2) = \ldots \longleftarrow \operatorname{Hom}(\mathbb{Z}/4,\mathbb{Z}/2) \longleftarrow \ldots \longleftarrow \operatorname{Hom}(\mathbb{Z}/4,\mathbb{Z}/2) \longleftarrow 0.$$

Notice that  $\operatorname{Hom}(\mathbb{Z}/4, \mathbb{Z}/2) \cong \mathbb{Z}/2$  since to define a homomorphism from  $\mathbb{Z}/4$  to  $\mathbb{Z}/2$ , a generator of  $\mathbb{Z}/4$  can either be mapped to  $0 + \mathbb{Z}/2$  or to  $1 + \mathbb{Z}/2$ . The homomorphisms in the long exact sequence are given by the dual of multiplication by 2, which is the zero map in the depicted case. Hence, we obtain

 $\mathrm{H}^{n}(\mathrm{Hom}(F^{\mathbb{Z}/2},\mathbb{Z}/2)) \cong \ker(\cdot 0)/\operatorname{im}(\cdot 0) \cong (\mathbb{Z}/2)/0 \cong \mathbb{Z}/2 \text{ for all } n \ge 0.$ This implies that

$$\operatorname{Ext}_{\mathbb{Z}/4}^{n}(\mathbb{Z}/2,\mathbb{Z}/2)\cong\mathbb{Z}/2$$

for all  $n \ge 0$ .

## Problem 6

#### Maria Morariu

**a).** Let p denote the given covering of  $S^1$ . We start by showing that  $\tilde{\sigma}(1) - \tilde{\sigma}(0)$  does not depend on the choice of the lift  $\tilde{\sigma}$ . Let  $\bar{\sigma} : [0,1] \to \mathbb{R}$  be a further lift of  $\sigma$ . Define  $\bar{\sigma}' : [0,1] \to \mathbb{R}$ ,  $\bar{\sigma}'(t) = \tilde{\sigma}(t) + \bar{\sigma}(0) - \tilde{\sigma}(0)$ . This map is continuous with  $\bar{\sigma}'(0) = \bar{\sigma}(0)$  and  $e^{2\pi i \bar{\sigma}'(t)} = e^{2\pi i \bar{\sigma}(t)} e^{2\pi i \bar{\sigma}(0)} \left(e^{2\pi i \bar{\sigma}(0)}\right)^{-1} = \sigma(t)\sigma(0)\sigma(0)^{-1} = \sigma(t)$ , so  $\bar{\sigma}'$  is a lift of  $\sigma$  with  $\bar{\sigma}'(0) = \bar{\sigma}(0)$ . By the uniqueness in the lifting property of covers, it follows  $\bar{\sigma}' = \bar{\sigma}$  and in particular  $\bar{\sigma}(1) = \bar{\sigma}'(1) = \tilde{\sigma}(1) + \bar{\sigma}(0) - \tilde{\sigma}(0)$  and thus  $\bar{\sigma}(1) - \bar{\sigma}(0) = \tilde{\sigma}(1) - \tilde{\sigma}(0)$ . Therefore,  $\tilde{\sigma}(1) - \tilde{\sigma}(0)$  does not depend on the choice of the lift  $\tilde{\sigma}$  and we can define the map  $\phi \colon C_1(S^1) \to \mathbb{R}$  as the linear map with  $\phi(\sigma) = \tilde{\sigma}(1) - \tilde{\sigma}(0)$  for any 1-simplex  $\sigma$ . By definition, this is a 1-cochain of  $S^1$  with coefficients in  $\mathbb{R}$ .

Let us show that  $\phi$  is actually a 1-cocycle. By Remark 5 in the lecture, this is the same as showing that  $\phi$  is 0 on 1-boundaries. Let  $\sigma: \Delta^2 \to S^1$  be a singular 2-simplex. We show that  $\phi(d\sigma) = 0$ . By definition  $d\sigma = \sigma|_{[1,2]} - \sigma|_{[0,2]} + \sigma|_{[0,1]}$ . Since  $\Delta^2$  is simply connected, the lifting property of covers implies that there exists a lift  $\tilde{\sigma}: \Delta^2 \to \mathbb{R}$  such that  $p \circ \tilde{\sigma} = \sigma$ . Then  $\tilde{\sigma}|_{[1,2]}, \tilde{\sigma}|_{[0,2]}, \tilde{\sigma}|_{[0,1]}$  are lifts of  $\sigma|_{[1,2]}, \sigma|_{[0,2]}, \sigma|_{[0,1]}$ . Hence, we have

$$\begin{split} \phi(d\sigma) &= \phi(\sigma|_{[1,2]}) - \phi(\sigma|_{[0,2]}) + \phi(\sigma|_{[0,1]}) \\ &= \tilde{\sigma}|_{[1,2]}(1) - \tilde{\sigma}|_{[1,2]}(0) - \left(\tilde{\sigma}|_{[0,2]}(1) - \tilde{\sigma}|_{[0,2]}(0)\right) + \tilde{\sigma}|_{[0,1]}(1) - \tilde{\sigma}|_{[0,1]}(0) \\ &= \tilde{\sigma}(2) - \tilde{\sigma}(1) - \tilde{\sigma}(2) + \tilde{\sigma}(0) + \tilde{\sigma}(1) - \tilde{\sigma}(0) = 0. \end{split}$$

Since all 1-boundaries can be written as finite sums of such  $d\sigma$ , it follows that  $\phi$  is zero on 1-boundaries, so  $\phi$  is a 1-cocycle.

Lastly, we show that  $\phi$  generates  $\mathrm{H}^1(S^1, \mathbb{R})$ . Note that  $\mathrm{H}_0(S^1) \cong \mathbb{Z}$  and it is in particular free, so  $\mathrm{Ext}(\mathrm{H}_0(S^1), \mathbb{R}) \cong 0$  and by the universal coefficient theorem for cohomology, it follows that evaluation map  $ev \colon \mathrm{H}^1(X; \mathbb{R}) \to \mathrm{Hom}(\mathrm{H}_1(S^1), \mathbb{R}), [\psi] \mapsto \psi$ is an isomorphism. Also,  $\mathrm{H}_1(S^1) \cong \mathbb{Z}$ , so we have a natural isomorphism  $\mathrm{Hom}(\mathrm{H}_1(S^1), \mathbb{R}) \to \mathbb{R}, \psi \mapsto \psi([\sigma])$ , where  $\sigma \colon [0, 1] \to S^1, t \mapsto e^{2\pi i t}$ . Let us note that  $\tilde{\sigma} \colon [0, 1] \to \mathbb{R}, t \mapsto t$  is a lift of  $\sigma$ , so  $\phi(\sigma) = \tilde{\sigma}(1) - \tilde{\sigma}(0) = 1 - 0 = 1$ , which is a generator for  $\mathbb{R}$ , so  $[\phi]$  is a generator for  $\mathrm{H}^1(S^1, \mathbb{R})$ .

**b).** Again, we start by showing that for any path  $\sigma: [0,1] \to S^1$  and any lift  $\tilde{\sigma}$  of  $\sigma$ , the expression  $\lfloor \tilde{\sigma}(1) \rfloor - \lfloor \tilde{\sigma}(0) \rfloor$  does not depend on the choice of the lift. Let  $\tilde{\sigma}$  and  $\bar{\sigma}$  be two lifts of  $\sigma$ . In a), we have seen that

 $\bar{\sigma}(1) - \bar{\sigma}(0) = \tilde{\sigma}(1) - \tilde{\sigma}(0)$  and thus  $\tilde{\sigma}(1) - \bar{\sigma}(1) = \tilde{\sigma}(0) - \bar{\sigma}(0)$ . For any  $t \in [0, 1]$ , we have  $e^{2\pi i \bar{\sigma}(t)} = \sigma(t) e^{2\pi i \bar{\sigma}(t)}$ , so  $\tilde{\sigma} - \bar{\sigma} \in \mathbb{Z}$  and therefore  $\{\tilde{\sigma}(t)\} = \{\bar{\sigma}(t)\}$ , where

by 
$$\{x\} = x - \lfloor x \rfloor$$
 we denote the fractional part of  $x \in \mathbb{R}$ . Hence,  $\tilde{\sigma}(t) - \bar{\sigma}(t) = \lfloor \tilde{\sigma}(t) \rfloor + \{\tilde{\sigma}(t)\} - \lfloor \bar{\sigma}(t) \rfloor - \{\bar{\sigma}(t)\} = \lfloor \tilde{\sigma}(t) \rfloor - \lfloor \bar{\sigma}(t) \rfloor$ . In particular,

 $\begin{bmatrix} \tilde{\sigma}(1) \end{bmatrix} - \begin{bmatrix} \bar{\sigma}(1) \end{bmatrix} = \tilde{\sigma}(1) - \bar{\sigma}(1) = \tilde{\sigma}(0) - \bar{\sigma}(0) = \begin{bmatrix} \tilde{\sigma}(0) \end{bmatrix} - \begin{bmatrix} \bar{\sigma}(0) \end{bmatrix}.$ 

Thus, we can define the map  $\phi: C_1(S^1) \to \mathbb{Z}$  by linearly extending  $\phi(\sigma) = \lfloor \tilde{\sigma}(1) \rfloor - \lfloor \tilde{\sigma}(0) \rfloor$ .

The fact that  $\phi$  is a cocycle follows exactly as in a), since the floors of two equal numbers are equal. Also, as in a), we get a natural isomorphism  $H^1(S^1, \mathbb{Z}) \to \mathbb{Z}, [\psi] \mapsto \psi(\sigma)$ , where  $\sigma \colon [0, 1] \to S^1, t \mapsto e^{2\pi i t}$ . Observe that  $[\phi]$  is mapped to 1 under this isomorphism, which is a generator for  $\mathbb{Z}$  and thus  $[\phi]$  is a generator for  $H^1(S^1, \mathbb{Z})$ .

### Problem 7

## Clara Bonvin

The UCT for cohomology gives the following SES :

$$0 \longrightarrow \operatorname{Ext}(H_0(X,\mathbb{Z}),\mathbb{Z}) \longrightarrow H^1(X;\mathbb{Z}) \longrightarrow \operatorname{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z}) \longrightarrow 0$$

Note that  $H_0(X, \mathbb{Z})$  is a free  $\mathbb{Z}$  Module, therefore we have  $\text{Ext}(H_0(X, \mathbb{Z}), \mathbb{Z}) = 0$ . From the above SES, we get :  $H^1(X, \mathbb{Z}) \cong \text{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z})$ .

Therefore, it suffices to show that  $\text{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z})$  is torsion free to deduce that  $H^1(X,\mathbb{Z})$  is torsion free as well.

To show that  $\operatorname{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z})$  is torsion free, we show that  $\operatorname{Hom}(M,\mathbb{Z})$  is torsion free for any  $\mathbb{Z}$  Module M. We consider its torsion group  $\operatorname{T}(\operatorname{Hom}(M,\mathbb{Z})) = \{\varphi \in \operatorname{Hom}(M,\mathbb{Z}) : \exists \lambda \in \mathbb{Z} \setminus \{0\} \text{ with } \lambda \varphi = 0\}$  and show that it is trivial.

Let  $\varphi \in T(Hom(M,\mathbb{Z}))$ , then  $\forall m \in M$ , there exists some  $\lambda \neq 0$  such that  $\lambda \varphi(m) = 0$ .

This gives  $\forall m \in M : \varphi(m) = 0 \Rightarrow \varphi = 0$  and therefore we get  $T(Hom(M, \mathbb{Z})) = 0$ .

Solutions Sheet 3

Problem 8

Sina Keller and Tristan Lovsin

Since b) and c) are more intuitive to understand we put subpart a) at the end.

b).

Cellular homology. We look at the Klein bottle  $K^2$  as the square with the edges identified as shown in the following picture:



We then have the following cochains in cohomology:

$$\begin{array}{lll} 0 - \text{cochains:} & \varphi : x \mapsto 1 \\ 1 - \text{cochains:} & \alpha : \begin{cases} a & \mapsto 1 \\ b & \mapsto 0 \end{cases} \text{ and } \beta : \begin{cases} a & \mapsto 0 \\ b & \mapsto 1 \end{cases} \\ 2 - \text{cochains:} & \gamma : A \mapsto 1 \end{array}$$

These are maps from the cellular chain group to  $\mathbb{Z}$  or  $\mathbb{F}_2$  respectively and are generating  $C^i(T^2; \mathbb{Z} \text{ or } \mathbb{F}_2)$  for i = 0, 1, 2. Now let's look at the boundary maps.

$$d^{0}: C^{0} \to C^{1}$$
$$\varphi \mapsto d^{0}(\varphi)$$
$$d^{0}(\varphi)(a) = \varphi(d_{1}(a)) = \varphi(x - x) = 0$$
$$d^{0}(\varphi)(b) = \varphi(d_{1}(b)) = \varphi(x - x) = 0$$

From this we get the kernel and image of  $d^0$ :

(1) 
$$\ker(d^0) = \langle \varphi \rangle$$

 $(2) \qquad \qquad \operatorname{im}(d^0) = 0$ 

Solutions Sheet 3

Now we do the same for  $d^1$ :

$$d^{1}: C^{1} \to C^{2}$$
  

$$\alpha \mapsto d^{1}(\alpha)$$
  

$$d^{1}(\alpha)(A) = \alpha(d_{2}(A)) = \alpha(b) - \alpha(a) + \alpha(b) + \alpha(a) = 0 - 1 + 0 + 1 = 0$$
  

$$d^{1}(\beta)(A) = \beta(d_{2}(A)) = \beta(b) - \beta(a) + \beta(b) + \beta(a) = 2\beta(b) = 2 \implies d^{1}(\beta) = 2\gamma$$

Here we get two different cases for coefficients in  $\mathbb{Z}$  and  $\mathbb{F}_2$ .

$$\mathbb{Z} \qquad \mathbb{F}_2$$
(3) 
$$\ker(d^1) = \langle \phi \rangle \qquad \ker(d^1) = \langle \phi, \beta \rangle$$

(3) 
$$\ker(d^{1}) = \langle \alpha \rangle$$
  $\ker(d^{1}) = \langle \alpha, \beta \rangle$ 

(4) 
$$\operatorname{im}(d^1) = \langle 2\gamma \rangle \qquad \operatorname{im}(d^1) = 0$$

Now let's do the same for  $d^2$ :

$$d^{2}: C^{2} \to C^{3}$$
$$\gamma \mapsto d^{2}(\gamma)$$
$$d^{2}(\gamma)(0) = \gamma(d_{3}(0)) = 0$$

We get for both  $\mathbb{Z}$  and  $\mathbb{F}_2$  that:

(5) 
$$\ker(d^2) = \langle \gamma \rangle$$
$$\operatorname{im}(d^2) = 0$$

Now we combine all these equations to get the *i*-th cohomology with coefficient in  $\mathbb{Z}$  and  $\mathbb{F}_2$ :

$$\mathbb{Z} \qquad \mathbb{F}_{2}$$

$$H^{0} \cong \operatorname{ker}(d^{0})_{\operatorname{im}(d^{-1})} \stackrel{(1)}{\cong} \qquad \mathbb{Z}_{0} = \mathbb{Z} \qquad \mathbb{F}_{2}_{0} = \mathbb{F}_{2}$$

$$H^{1} \cong \operatorname{ker}(d^{1})_{\operatorname{im}(d^{0})} \stackrel{(2),(3)}{\cong} \qquad \mathbb{Z}_{0} = \mathbb{Z} \qquad \mathbb{F}_{2}^{2}_{0} = \mathbb{F}_{2}^{2}$$

$$H^{2} \cong \operatorname{ker}(d^{2})_{\operatorname{im}(d^{1})} \stackrel{(4),(5)}{\cong} \qquad \mathbb{Z}_{2\mathbb{Z}} = \mathbb{F}_{2} \qquad \mathbb{F}_{2}_{0} = \mathbb{F}_{2}$$

Checking with UCT. Now let's check if this result is the same as in the UCT of cohomology. We check the following SES, where A is either  $\mathbb{Z}$  or  $\mathbb{F}_2$  and C our chain complex of  $K^2$ :

$$0 \to \operatorname{Ext}(H_{i-1}(C), A) \to H^i(C; A) \to \operatorname{Hom}(H_i(C), A) \to 0$$

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We have to check the following SES:

(6) 
$$0 \to \operatorname{Ext}(H_{-1}(K^2), \mathbb{Z}) \to H^0(K^2; \mathbb{Z}) \to \operatorname{Hom}(H_0(K^2), \mathbb{Z}) \to 0$$
  
(7) 
$$0 \to \operatorname{Ext}(H_0(K^2), \mathbb{Z}) \to H^1(K^2; \mathbb{Z}) \to \operatorname{Hom}(H_1(K^2), \mathbb{Z}) \to 0$$

(8) 
$$0 \to \operatorname{Ext}(H_1(K^2), \mathbb{Z}) \to H^2(K^2; \mathbb{Z}) \to \operatorname{Hom}(H_2(K^2), \mathbb{Z}) \to 0$$

(9) 
$$0 \to \operatorname{Ext}(H_{-1}(K^2), \mathbb{F}_2) \to H^0(K^2; \mathbb{F}_2) \to \operatorname{Hom}(H_0(K^2), \mathbb{F}_2) \to 0$$

(10) 
$$0 \to \operatorname{Ext}(H_0(K^2), \mathbb{F}_2) \to H^1(K^2; \mathbb{F}_2) \to \operatorname{Hom}(H_1(K^2), \mathbb{F}_2) \to 0$$

(11) 
$$0 \to \operatorname{Ext}(H_1(K^2), \mathbb{F}_2) \to H^2(K^2; \mathbb{F}_2) \to \operatorname{Hom}(H_2(K^2), \mathbb{F}_2) \to 0$$

From AlgTopo I we remember the homology groups of  $K^2$ :

(12) 
$$H_i \cong \begin{cases} \mathbb{Z} & i=0\\ \mathbb{Z} \oplus \mathbb{F}_2 & i=1\\ 0 & else \end{cases}$$

We know from Prop 8, that if A is free  $Ext(A, B) \cong 0$ .

0th cohomology. Therefore (6) and (9) with help of (12) becomes the following as desired:

(13)  

$$(6): 0 \to H^{0}(K^{2}; \mathbb{Z}) \to \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \to 0$$

$$\Rightarrow H^{0}(K^{2}; \mathbb{Z}) \cong \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$$

$$(9): 0 \to H^{0}(K^{2}; \mathbb{F}_{2}) \to \operatorname{Hom}(\mathbb{Z}, \mathbb{F}_{2}) \to 0$$

(14) 
$$\Rightarrow H^0(K^2; \mathbb{F}_2) \cong \operatorname{Hom}(\mathbb{Z}, \mathbb{F}_2) \cong \mathbb{F}_2$$

1st cohomology. Similarly (7) and (10) with (12) and the fact that  $\mathbb{Z}$  is free turns into the following, as desired:

$$(7): 0 \to H^{1}(K^{2}; \mathbb{Z}) \to \operatorname{Hom}(\mathbb{Z} \oplus \mathbb{F}_{2}, \mathbb{Z}) \to 0$$
  

$$\Rightarrow H^{1}(K^{2}; \mathbb{Z}) \cong \operatorname{Hom}(\mathbb{Z} \oplus \mathbb{F}_{2}, \mathbb{Z}) \cong \mathbb{Z}$$
  

$$(10): 0 \to H^{1}(K^{2}; \mathbb{F}_{2}) \to \operatorname{Hom}(\mathbb{Z} \oplus \mathbb{F}_{2}, \mathbb{F}_{2}) \to 0$$
  

$$\Rightarrow H^{1}(K^{2}; \mathbb{F}_{2}) \cong \operatorname{Hom}(\mathbb{Z} \oplus \mathbb{F}_{2}, \mathbb{F}_{2}) \stackrel{*}{\cong} \mathbb{F}_{2}^{2}$$

\* is deduced from the fact that  $\operatorname{Hom}(A\oplus B,C)\cong\operatorname{Hom}(A,C)\oplus\operatorname{Hom}(B,C)$  which in this case yields

$$\operatorname{Hom}(\mathbb{Z} \oplus \mathbb{F}_2, \mathbb{F}_2) \cong \operatorname{Hom}(\mathbb{Z}, \mathbb{F}_2) \oplus \operatorname{Hom}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$$

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2nd cohomology. Now we need the fact that  $\text{Ext}(A \oplus B, C) \cong \text{Ext}(A, C) \oplus \text{Ext}(B, C)$ from Prop 8. This turns (8) and (11) with help of (12) into:

$$(8): 0 \to \operatorname{Ext}(\mathbb{Z}, \mathbb{Z}) \oplus \operatorname{Ext}(\mathbb{F}_2, \mathbb{Z}) \to H^2(K^2; \mathbb{Z}) \to 0$$

$$(15) \quad \operatorname{Ext}(\mathbb{F}_2, \mathbb{Z}) \cong 0 \oplus \operatorname{Ext}(\mathbb{F}_2, \mathbb{Z}) \cong \operatorname{Ext}(\mathbb{Z}, \mathbb{Z}) \oplus \operatorname{Ext}(\mathbb{F}_2, \mathbb{Z}) \cong H^2(K^2; \mathbb{Z})$$

$$(11): 0 \to \operatorname{Ext}(\mathbb{Z}, \mathbb{F}_2) \oplus \operatorname{Ext}(\mathbb{F}_2, \mathbb{F}_2) \to H^2(K^2; \mathbb{F}_2) \to 0$$

(16)

 $\operatorname{Ext}(\mathbb{F}_2,\mathbb{F}_2) \cong 0 \oplus \operatorname{Ext}(\mathbb{F}_2,\mathbb{F}_2) \cong \operatorname{Ext}(\mathbb{Z},\mathbb{F}_2) \oplus \operatorname{Ext}(\mathbb{F}_2,\mathbb{F}_2) \cong H^2(K^2;\mathbb{F}_2)$ 

Now we want to calculate Ext for (15) and (16). We remember the definition of Ext from the lecture:

$$\operatorname{Ext}(M, N) \coloneqq H^1(\operatorname{Hom}(F^M, N))$$

Therefore we try to calculate the following:

(17) 
$$\operatorname{Ext}(\mathbb{F}_2, \mathbb{Z}) = H^1(\operatorname{Hom}(F^{\mathbb{F}_2}, \mathbb{Z}))$$

Let  $F^{\mathbb{F}_2} := 0 \to \mathbb{Z} \xrightarrow{\cdot^2} \mathbb{Z} \to 0$  be a free resolution of  $\mathbb{F}_2$ . Then by definition  $\operatorname{Hom}(F^{\mathbb{F}_2},\mathbb{Z})$  form the corresponding cochain complexes and we calculate  $\operatorname{im}(d^0)$  and  $\operatorname{ker}(d^1)$ . Let  $\sigma, \tau$  be an arbitrary element in  $C^0$ ,  $C^1$  respectively. Let a, b be elements that were defined in part 1.

$$d^{0}: \quad C^{0} \to C^{1}$$
  

$$\sigma \mapsto d^{0}(\sigma)$$
  

$$d^{0}(\sigma)(b) = \sigma(d_{1}(b)) = \sigma(2b) = 2$$
  

$$d^{1}: \quad C^{1} \to C^{2}$$
  

$$\tau \mapsto d^{1}(\tau)$$
  

$$d^{1}(\tau)(0) = 0$$

Therefore we have that  $\operatorname{im}(d^0) \cong 2\mathbb{Z}$  and  $\operatorname{ker}(d^1) \cong \mathbb{Z}$ . Now we have that

(18)  
$$H^{1} \cong \overset{\operatorname{ker}(d^{1})}{\underset{\operatorname{im}(d^{0})}{\cong}} \cong \mathbb{Z}_{2\mathbb{Z}} \cong \mathbb{F}_{2}$$
$$\overset{(17)}{\cong} \operatorname{Ext}(\mathbb{F}_{2}, \mathbb{Z})$$

This is exactly the result we expected from our calculations from cellular cohomology.

Now onto (16) where we try to calculate the following:

(19) 
$$\operatorname{Ext}(\mathbb{F}_2, \mathbb{F}_2) = H^1(\operatorname{Hom}(F^{\mathbb{F}_2}, \mathbb{F}_2))$$

We have that

 $\operatorname{Hom}(\mathbb{Z}, \mathbb{F}_2) \cong \mathbb{F}_2$ 

and

$$d^{0}: \quad C^{0} \to C^{1}$$
  

$$\sigma \mapsto d^{0}(\sigma)$$
  

$$d^{0}(\sigma)(b) = \sigma(d_{1}(b)) = \sigma(2b) = 2\sigma \equiv 0$$
  

$$d^{1}: \quad C^{1} \to C^{2}$$
  

$$\tau \mapsto d^{1}(\tau)$$
  

$$d^{1}(\tau)(0) = 0$$

Therefore we have that  $\operatorname{im}(d^0) \cong 2\mathbb{F}_2 \cong 0$  and  $\operatorname{ker}(d^1) \cong \mathbb{F}_2$ , thus

(20)  
$$H^{1} \cong \overset{\operatorname{ker}(d^{1})}{\underset{\operatorname{im}(d^{0})}{\cong}} \cong \overset{\mathbb{F}_{2}}{\underset{\operatorname{Ext}(\mathbb{F}_{2}, \mathbb{F}_{2})}{\overset{(19)}{\cong}} \operatorname{Ext}(\mathbb{F}_{2}, \mathbb{F}_{2})$$

Therefore all our results align with UCT and we are done.

c).

Cellular cohomology. We know from AlgTopo I that  $\mathbb{R}P^n$  has the following cellular homology:

$$n - \text{cell}: c_n$$

$$n - 1 - \text{cell}: c_{n-1}$$

$$\vdots \vdots$$

$$0 - \text{cell}: c_0$$

With the following boundary maps:

(21) 
$$d_i(c_i) = \begin{cases} 0 & \text{i odd} \\ 2c_{i-1} & \text{i even} \end{cases}$$

Therefore we construct the following cellular cohomology:

$$n - \operatorname{cell}: \quad \varphi_n : c_n \mapsto 1$$

$$n - 1 - \operatorname{cell}: \quad \varphi_{n-1} : c_{n-1} \mapsto 1$$

$$\vdots \quad \vdots$$

$$0 - \operatorname{cell}: \quad \varphi_0 : c_0 \mapsto 1$$

Now we look at the boundary maps in cohomology:

$$d^{i}(\varphi_{i})(c_{i+1}) = \varphi_{i}(d_{i+1}(c_{i+1})) \stackrel{(21)}{=} \begin{cases} 2\varphi_{i} & \text{i odd} \\ 0 & \text{i even} \end{cases}$$

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This gives us the kernel and the image:

$$\mathbb{Z} \qquad \mathbb{F}_{2}$$
(22) 
$$\ker(d^{i}) = \begin{cases} 0 & \text{i odd} \\ \langle \varphi_{i} \rangle & \text{i even} \end{cases} \qquad \ker(d^{i}) = \begin{cases} \langle \varphi_{i} \rangle & \text{i odd} \\ \langle \varphi_{i} \rangle & \text{i even} \end{cases}$$
(23) 
$$\operatorname{im}(d^{i}) = \begin{cases} \langle 2\varphi_{i} \rangle & \text{i odd} \\ 0 & \text{i even} \end{cases} \qquad \operatorname{im}(d^{i}) = \begin{cases} 0 & \text{i odd} \\ 0 & \text{i even} \end{cases}$$

Now we can put this together to get the cohomology groups with coefficient in A which is  $\mathbb{Z}$  or  $\mathbb{F}_2$  respectively:<sup>1</sup>

$$H^{i}(\mathbb{R}P^{n}; A) \cong \ker(d^{i})_{im(d^{i-1})} \stackrel{(22),(23)}{\cong} \begin{cases} \mathbb{Z}: & \mathbb{F}_{2}: \\ \mathbb{Z}_{0} \cong \mathbb{Z} & \mathbb{F}_{2}_{0} \cong \mathbb{F}_{2} & i = n \text{ with } n \text{ odd } \text{ or } i = 0 \\ 0_{0} \cong 0 & \mathbb{F}_{2}_{0} \cong \mathbb{F}_{2} & 0 < i < n, i \text{ odd} \\ \mathbb{Z}_{2\mathbb{Z}} \cong \mathbb{F}_{2} & \mathbb{F}_{2}_{0} \cong \mathbb{F}_{2} & 0 < i \leq n, i \text{ even} \end{cases}$$

Checking with UCT. Now we have to check if those results are in alignment with UCT. Again we have that A is either  $\mathbb{Z}$  or  $\mathbb{F}_2$  and C our chain complex of  $\mathbb{R}P^n$ 

$$0 \to \operatorname{Ext}(H_{i-1}(C), A) \to H^i(C; A) \to \operatorname{Hom}(H_i(C), A) \to 0$$

We remember the homology group of  $\mathbb{R}P^2$  from AlgTopo I:

(24) 
$$H_i(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & i=0, i=n \text{ and } n \text{ odd} \\ \mathbb{F}_2 & i \text{ odd and } 0 < i < n \\ 0 & \text{else} \end{cases}$$

We get now the following different cases for the SES with help of Prop 8:

(25) 
$$0 \to H^i(\mathbb{R}P^n; A) \to \operatorname{Hom}(\mathbb{Z}, A) \to 0 \quad i = 0, i = n \text{ and } n \text{ odd}$$

(26) 
$$0 \to H^i(\mathbb{R}P^n; A) \to \operatorname{Hom}(\mathbb{F}_2, A) \to 0$$
 *i* odd and  $0 < i < n$ 

(27) 
$$0 \to \operatorname{Ext}(\mathbb{F}_n, A) \to H^i(\mathbb{R}P^2; A) \to 0$$
 *i* even and  $0 < i < n$ 

We have already calculated in (13) and (14) what  $\operatorname{Hom}(\mathbb{Z}, A)$  is for both cases. Therefore (25) aligns with UCT.

We have calculated  $\text{Ext}(\mathbb{F}_n, A)$  in (18) and (20) for both cases. Therefore (27) aligns with UCT as well.

Now let's look at both cases for (26). In the case of  $A = \mathbb{Z}$  we have that  $\operatorname{Hom}(\mathbb{F}_2, \mathbb{Z}) \cong 0$  and when  $A = \mathbb{F}_2$  we have  $\operatorname{Hom}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2$ , yielding the desired result for all cases and we are done.

<sup>&</sup>lt;sup>1</sup>Correction 27 July 2024: Previously, the next line said  $H^i(K^2; A)$ .

a).

Cellular homology. We view  $T^n = {I^n} / \sim$  as the *n*-dim cube with the equivalence relation  $\sim$  which identifies opposite facets of the boundary.

We construct our cells similarly to a hypercube without identifying opposite facets. As Wikipedia stated, we imagine the hypercube in a Cartesian coordinate system. Then there exists for every *i*-dim cell *i* coordinate axes that are parallel to this element. This results in  $\binom{n}{k}$  elements. Differently to Wikipedia, we don't have to multiply by  $2^{n-i}$  for the total amount of cells, because the parallel cells on the other side of the hypercube are identified.<sup>2</sup>

So we have  $\binom{n}{i}$  *i*-cells for each *i*-dim cochain group  $C_i$ . We call them  $c_k^i$  for  $i = 0, \ldots, n$  and  $k = 0, \ldots, \binom{n}{i}$ . Now we compute the boundary maps  $d_i$ .

Quick reminder how cellular homology works. We look at embeddings f of the n-dimensional ball  $B^n$  and its boundary  $\partial B^n$  into the  $K^{(n)}$  and  $K^{(n-1)}$  skeletons respectively, where these maps are injective on the interior of  $B^n$  but not necessarily on the boundary. Then we took the quotient space  $K^{(n)}/K^{(n-1)} \simeq \bigvee S^n$  and looked at the projection onto one of those spheres. This projection we called  $p^3$ . In this exercise we will use the notation from AlgTopo I and the solution of exercise 3 of the exercise sheet 7.

For  $d_i$  we consider any of the maps  $p_{c_k^{i-1}}f_{\partial c_m^i}: \partial I^i \to S^{i-1}$ . We note that there are two opposite facets of  $I^i$  in whose interiors this map restricts to a homeomorphism. The map collapses the rest of  $\partial I^i$  to a point in  $S^{i-1}$ . The degree of  $p_{c_k^{i-1}}f_{\partial c_m^i}$  is therefore the sum of the two local degrees at any two points in  $q_1, q_2$  in the two first-mentioned facets which get mapped to the same point in  $T^n$ . Now we note that the restriction of  $p_{c_k^{i-1}}f_{\partial c_m^i}$  to these faces are obtained from one another by precomposing with an orientation-reversing map. Therefore the sum of these local degrees vanishes. Therefore we have that  $\deg(d_i) = 0$  for all i.<sup>4</sup>

For the cellular cohomology we take the cochains  $C^i := \text{Hom}(C_i, A)$ , for A either  $\mathbb{Z}$  or  $\mathbb{F}_2$ . We again have  $\binom{n}{i}$  *i*-cochains for each *i*-dim cochain group  $C^i$ .

Now we want to calculate the coboundary maps  $d^i$ . We know that the coboundary maps are the transpose of the boundary maps and since the boundary maps are all 0, we have that  $d^i = 0$  for all *i* as well.

We have that  $\ker(d^i) \cong A^{\binom{n}{i}}$  and  $\operatorname{im}(d^i) \cong 0$  for all *i*. We immediately get the cohomology groups:

$$H^{i}(T^{n};\mathbb{Z}) \cong \overset{\operatorname{ker}(d^{i})}{\underset{\operatorname{im}(d^{i-1})}{\cong}} \cong \overset{\mathbb{Z}\binom{n}{i}}{\underset{0}{\cong}}_{0} \cong \mathbb{Z}\binom{n}{i}$$
$$H^{i}(T^{n};\mathbb{F}_{2}) \cong \overset{\operatorname{ker}(d^{i})}{\underset{\operatorname{im}(d^{i-1})}{\cong}} \cong \overset{\mathbb{F}_{2}^{\binom{n}{i}}}{\underset{0}{\cong}}_{0} \cong \mathbb{F}_{2}^{\binom{n}{i}}$$

<sup>2</sup>Wikipedia: Hyperwüfel

 $<sup>^{3}</sup>$ A detailed explanation of this is found in the lecture notes of Lecture 25 AlgTopo I starting at page 6.

<sup>&</sup>lt;sup>4</sup>Generalized version of proof for exercise 3 on exercise sheet 7 in AlgTopo I

Checking with UCT. Now we check if this result aligns with UCT for cohomology: (28)  $0 \to \text{Ext}(H_{i-1}(C), A) \to H^i(C; A) \to \text{Hom}(H_i(C), A) \to 0$ 

We know that  $H_i(T^n) = \mathbb{Z}^{\binom{n}{i}}$  from AlgTopo I. This is just a finite number of copies of  $\mathbb{Z}$  and therefore free. We know from Prop 8, that if A is free  $\text{Ext}(A, B) \cong 0$ . We note that  $\text{Hom}(\mathbb{Z}^i, \mathbb{Z}) \cong \mathbb{Z}^i$  and  $\text{Hom}(\mathbb{Z}^i, \mathbb{F}_2) \cong \mathbb{F}_2^i$ . Therefore (28) in our case becomes:

$$0 \to H^{i}(T^{n}; A) \to \operatorname{Hom}(\mathbb{Z}^{\binom{n}{i}}, A) \to 0$$
  
$$\Rightarrow 0 \to H^{i}(T^{n}; A) \to A^{\binom{n}{i}} \to 0$$
  
$$\Rightarrow H^{i}(T^{n}; \mathbb{Z}) \cong \mathbb{Z}^{\binom{n}{i}} \text{ and}$$
  
$$H^{i}(T^{n}; \mathbb{F}_{2}) \cong \mathbb{F}_{2}^{\binom{n}{i}}$$

This is exactly what we expect from cellular cohomology and we are done.

## Problem 9

no solutions for starred problems