Problem 1

Leon Dahlmeier

We want to show $\mathrm{Tor}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/g\mathbb{Z}$ for $g := \gcd(m, n)$.
$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{d_0} \mathbb{Z}/n\mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}/n\mathbb{Z}$ called F. Where $\xrightarrow{n}$ is multiplication with n and d_0 the projection. Tensoring with $\mathbb{Z}/m\mathbb{Z}$ yields:

$$0 \to \mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} \xrightarrow{n\otimes id_{\mathbb{Z}/m\mathbb{Z}}} \mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} \xrightarrow{d_0\otimes id_{\mathbb{Z}/m\mathbb{Z}}} \mathbb{Z}/n\mathbb{Z} \to 0.$$

After simplifying everything we already know about the Tensor product:

$$0 \xrightarrow{0} \mathbb{Z}/m\mathbb{Z} \xrightarrow{n} \mathbb{Z}/m\mathbb{Z} \xrightarrow{d_0} \mathbb{Z}/n\mathbb{Z} \to 0$$

Therefore $\mathrm{Tor}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = H_1(F^{\mathbb{Z}/n\mathbb{Z}}, \mathbb{Z}/m\mathbb{Z}) = \ker(\mathbb{Z}/m\mathbb{Z} \xrightarrow{n} \mathbb{Z}/m\mathbb{Z})$

Finally, let us take a closer look at: $\ker(\mathbb{Z}/m\mathbb{Z} \xrightarrow{n} \mathbb{Z}/m\mathbb{Z})$. Remember $g = \gcd(m, n)$ and let u and k be such that $u \cdot g = m$ and $k \cdot g = n$. Since $n \cdot u = k \cdot u \cdot g = k \cdot m \equiv 0 \pmod{m}$, we have $\mathrm{im}(\mathbb{Z}/m\mathbb{Z} \xrightarrow{n} \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/u\mathbb{Z}$. We conclude using the isomorphism theorem:

$$\mathrm{Tor}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \frac{\mathbb{Z}/m\mathbb{Z}}{\mathbb{Z}/u\mathbb{Z}} \cong \mathbb{Z}/g\mathbb{Z}$$

Problem 2

Leon Dahlmeier

a). Since $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(T(A), T(B))$ let us assume without loss of generality that A and B are torsion. Let us define $C := \bigoplus_{0\neq b\in B} \mathbb{Z}/\operatorname{ord}(b)\mathbb{Z}$ and $f : C \to B$ by sending $[1] \in \mathbb{Z}/\operatorname{ord}(b)\mathbb{Z}$ to b.
The abelian group A together with the short exact sequence:

$$0 \to \ker(f) \to C \xrightarrow{f} B \to 0$$

give rise to the following long exact sequence:

$$0 \to \mathrm{Tor}(A, \ker(f)) \xrightarrow{g} \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, B) \to A \otimes \ker(f) \xrightarrow{h} A \otimes C \to A \otimes B \to 0$$

from which we can extract the short exact sequence:

$$0 \to \mathrm{coker}(g) \xrightarrow{\alpha} \mathrm{Tor}(A, B) \xrightarrow{\beta} \ker(h) \to 0$$

But by 4.14.

$$\mathrm{Tor}(A, C) = \bigoplus_{0\neq b\in B} \mathrm{Tor}(A, \mathbb{Z}/\operatorname{ord}(b)\mathbb{Z}) = \bigoplus_{0\neq b\in B} \ker(A \xrightarrow{\cdot\operatorname{ord}(b)} A)$$

which implies that $\mathrm{Tor}(A, C)$ as a subgroup of a torsion group is torsion. Also $A \otimes \ker(f)$ is torsion since we assumed A to be. Which then means that $\mathrm{coker}(g)$ and $\ker(h)$ are, meaning $\mathrm{Tor}(A, B)$ is:
For $x \in \mathrm{Tor}(A, B) \exists n \in \mathbb{N}$ s.t. $n\beta(x) = \beta(xn) = 0 \Rightarrow xn \in \ker(\beta) = \mathrm{im}(\alpha)$. Hence, $\exists y \in \mathrm{coker}(g)$ s.t. $\alpha(y) = xn$ but $\exists m \in \mathbb{N}$ s.t. $ym = 0$. Meaning $(xn)m = \alpha(y)m = \alpha(ym) = \alpha(0) = 0$, which concludes the proof.

b). The long exact sequence for $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ and the abelian group $T(A)$ is:

$$\begin{aligned} 0 \to \mathrm{Tor}(T(A), \mathbb{Z}) \to \mathrm{Tor}(T(A), \mathbb{Q}) \to \mathrm{Tor}(T(A), \mathbb{Q}/\mathbb{Z}) \to \\ T(A) \otimes \mathbb{Z} \to T(A) \otimes \mathbb{Q} \to T(A) \otimes \mathbb{Q}/\mathbb{Z} \to 0. \end{aligned}$$

$\mathbb{Z}$ and $\mathbb{Q}$ are torsion-free and $\mathbb{Z} \otimes T(A)$ is isomorphic to $T(A)$. Further for $a \otimes q \in T(A) \otimes \mathbb{Q}$ we have: $a \in T(A)$ meaning there is an $n \in \mathbb{Z} \setminus 0$ s.t. $a \cdot n = 0$ i.e. $a \otimes q = na \otimes \frac{q}{n} = 0$ implying $T(A) \otimes \mathbb{Q} \cong 0$. Using these isomorphisms we can simplify to:

$$0 \to 0 \to 0 \to \mathrm{Tor}(T(A), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\phi} T(A) \to T(A) \otimes \mathbb{Q} \to T(A) \otimes \mathbb{Q}/\mathbb{Z} \to 0$$

Which means that ϕ an isomorphism.

Problem 3

Aparna Jeyakumar

a). (a) From the Universal Coefficient Theorem, we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_n(X) \otimes M & \longrightarrow & H_n(X; M) & \longrightarrow & \mathrm{Tor}(H_{n-1}(X), M) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle f_* \otimes 1_M} & & \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle \mathrm{Tor}(f_*, 1_M)} & & \\ 0 & \longrightarrow & H_n(Y) \otimes M & \longrightarrow & H_n(Y; M) & \longrightarrow & \mathrm{Tor}(H_{n-1}(Y), M) & \longrightarrow & 0 \end{array}$$

Since $- \otimes 1_M$ and $\mathrm{Tor}(-, M)$ are additive functors between the category of abelian groups, they take isomorphisms to isomorphisms. In particular, $f_* \otimes 1_M$ and $\mathrm{Tor}(f_*, 1_M)$ are isomorphisms. Now, using the five lemma, we get that the vertical map in the middle $f_* : H_n(X; M) \to H_n(Y; M)$ is an isomorphism.

b). To show that $f_* : H_n(X) \to H_n(Y)$ is an isomorphism, it is enough to show that $H_n(Mc(f)) = 0$ for all n where $Mc(f)$ is the mapping cone complex of the map f. This is due to the following result from Homological Algebra:
If $f : X_\bullet \to Y_\bullet$ is a chain map of complexes then, the induced map on the homology, $f_* : H_n(X_\bullet) \to H_n(Y_\bullet)$ is an isomorphism iff $H_n(Mc(f)) = 0$ for all n, where $Mc(f)$ is the mapping cone complex of the map f.

We have that $f_* : H_n(X;\mathbb{Q}) \to H_n(Y,\mathbb{Q})$ is an isomorphism which implies that $H_n(Mc(f \otimes 1_{\mathbb{Q}})) = 0$ for all n. Since $\mathbb{Q}$ is torsion-free, $\mathrm{Tor}(H_n(Mc(f)),\mathbb{Q}) = 0$ and from the UCT for $Mc(f)$, we get that

$$H_n(Mc(f)) \otimes \mathbb{Q} \cong H_n(Mc(f);\mathbb{Q}) \cong H_n(Mc(f \otimes 1_{\mathbb{Q}})) \cong 0$$

(The second isomorphism is due to the distributive property of the tensor product over direct sums).
Similarly, we have $H_n(Mc(f \otimes 1_{\mathbb{Z}_p})) = 0$ for all p prime, for all n. Using the UCT again, we get that

$$(H_n(Mc(f)) \otimes \mathbb{Z}_p) \oplus \mathrm{Tor}(H_{n-1}(Mc(f)),\mathbb{Z}_p) \cong 0$$

which implies that both the terms are 0 and in particular, $\mathrm{Tor}(H_n(Mc(f)),\mathbb{Z}_p) = 0$ for all p prime and for all n. Setting $A = H_n(Mc(f))$, it is now enough to show that the following claim is true.

Claim : If A is an abelian group such that $A \otimes \mathbb{Q} = 0$ and $\mathrm{Tor}(A,\mathbb{Z}_p) = 0$ for all p prime, then $A = 0$.
Proof : Suppose $A \otimes \mathbb{Q} = 0$ and $\mathrm{Tor}(A,\mathbb{Z}_p) = 0$ for all p prime. Consider the short exact sequences

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \hookrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

Then, we get the following LESs,

$$0 \to \mathrm{Tor}(A,\mathbb{Z}) \to \mathrm{Tor}(A,\mathbb{Z}) \to \mathrm{Tor}(A,\mathbb{Z}_p) \to A \otimes \mathbb{Z} \xrightarrow{\cdot p} A \otimes \mathbb{Z} \to A \otimes \mathbb{Z}_p \to 0$$

$$0 \to \mathrm{Tor}(A,\mathbb{Z}) \to \mathrm{Tor}(A,\mathbb{Q}) \to \mathrm{Tor}(A,\mathbb{Q}/\mathbb{Z}) \to A \otimes \mathbb{Z} \to A \otimes \mathbb{Q} \to A \otimes Q/\mathbb{Z} \to 0$$

The first LES reduces to

$$0 \longrightarrow A \xrightarrow{\cdot p} A \longrightarrow A \otimes \mathbb{Z}_p \longrightarrow 0$$

The injectivity of the map $A \xrightarrow{\cdot p} A$ for all p implies that A is a torsion-free group. Then, $\mathrm{Tor}(A,\mathbb{Q}/\mathbb{Z}) \cong 0$ and the second LES reduces to $0 \to A \to 0$ and so $A = 0$.

Exercise 4. (Sheet 3)

(a) <u>Claim</u> For any group G, we have $\mathrm{Tor}(G, \mathbb{Q}) = 0$.

$\triangleright$ For any free resolution $0 \to F_1 \to F_0 \to G \to 0$ of G.

$F_1 \otimes \mathbb{Q} \longrightarrow F_0 \otimes \mathbb{Q}$ is a monomorphism. $\square$

$\Longrightarrow$ By UCT, $0 \to H_n(X;\mathbb{Z}) \otimes \mathbb{Q} \longrightarrow H_n(X;\mathbb{Q}) \to \underbrace{\mathrm{Tor}(H_{n-1}(X;\mathbb{Z}), \mathbb{Q})}_{=0} \to 0$

$\Longrightarrow H_n(X;\mathbb{Z}) \otimes \mathbb{Q} \cong H_n(X;\mathbb{Q})$.

(b) Easily follows from UCT.

Since $H_n(X;\mathbb{Z})$ is finitely generated,

$$H_n(X;\mathbb{Z}) \cong \mathbb{Z}^{\oplus k} \oplus \Big(\bigoplus_{\substack{q = \text{prime} \\ i_j \geq 1}} \mathbb{Z}/(q^{i_j}) \Big)$$

$$\Longrightarrow H_n(X;\mathbb{Z}) \otimes \mathbb{F}_p = \Big(\mathbb{Z}^{\oplus k} \oplus \Big(\bigoplus_{\substack{q = \text{prime} \\ i_j \geq 1}} \mathbb{Z}/(q^{i_j}) \Big) \Big) \otimes \mathbb{F}_p \cong$$

$$\cong (\mathbb{Z} \otimes \mathbb{F}_p)^{\oplus k} \oplus \Big(\bigoplus_{\substack{q = \text{prime} \\ i_j \geq 1}} \big[\mathbb{Z}/(q^{i_j}) \otimes \mathbb{F}_p\big] \Big)$$

if $q \neq p$ $\mathbb{Z}/(q^{i}) \otimes \mathbb{F}_p = 0$

$$\cong \mathbb{F}_p^{\oplus k} \oplus \Big(\bigoplus_{i_j \geq 1} \big[\mathbb{Z}/(p^{i_j}) \otimes \mathbb{F}_p\big] \Big) \cong \mathbb{F}_p^{\oplus k} \oplus \Big(\bigoplus \mathbb{F}_p \Big)$$

for each $\mathbb{Z}/(p^{i_j})$ in $H_n(X;\mathbb{F}_p)$.

Finally, $\mathbb{Z}/(p^{i}) \otimes \mathbb{F}_p \cong \mathbb{F}_p$.

Same holds for $\mathrm{Tor}(H_{n-1}(X;\mathbb{Z}); \mathbb{F}_p) = \mathrm{Tor}\Big(\mathbb{Z}^{\oplus k} \oplus \Big(\bigoplus_{\substack{q = \text{prime} \\ i_j \geq 1}} \mathbb{Z}/(q^{i_j}) \Big); \mathbb{F}_p\Big)$

$H_{n-1}(X;\mathbb{Z}) \cong \mathbb{Z}^{\oplus k} \oplus \Big(\bigoplus_{\substack{q = \text{prime} \\ i_j \geq 1}} \mathbb{Z}/(q^{i_j}) \Big)$

$$= \underbrace{\mathrm{Tor}(\mathbb{Z}^{\oplus k}; \mathbb{F}_p)}_{=0} \oplus \Big(\bigoplus_{\substack{q = \text{prime} \\ i_j \geq 1}} \mathrm{Tor}(\mathbb{Z}/(q^{i_j}), \mathbb{F}_p) \Big)$$

$$= \bigoplus_{\substack{q = \text{prime} \\ i_j \geq 1}} \mathbb{Z}/\gcd(q^{i_j}, p) \cong \bigoplus \mathbb{F}_p$$

for each $\mathbb{Z}/(p^{i})$ in $H_{n-1}(X;\mathbb{Z})$.

Problem 5

Naomi Rosenberg

We start by constructing a free resolution of $\mathbb{Z}/2$ as a $\mathbb{Z}/4$-module. To that extent, note that $\mathbb{Z}/2$ can be interpreted as $2+4\mathbb{Z}$, which is a submodule of $\mathbb{Z}/4$. Consider the following sequence:

$$\ldots \longrightarrow \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\text{proj}} \mathbb{Z}/2 \longrightarrow 0.$$

Note that $\ker(\cdot 2) = 2 + 4\mathbb{Z} = \operatorname{im}(\cdot 2)$ and $\ker(\text{proj}) = 2 + 4\mathbb{Z} = \operatorname{im}(\cdot 2)$. Consequently, the sequence defined above is a long exact sequence and therefore defines a free resolution F of the $\mathbb{Z}/4$-module $\mathbb{Z}/2$.

We thus get the following deleted free resolution:

$$F^{\mathbb{Z}/2} = \ldots \longrightarrow \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \longrightarrow 0.$$

This enables us to compute $\operatorname{Tor}_n^{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2)$. By plugging into the definition, we obtain:

$$\operatorname{Tor}_n^{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathrm{H}_n(F^{\mathbb{Z}/2}; \mathbb{Z}/2) = \mathrm{H}_n(F^{\mathbb{Z}/2} \otimes \mathbb{Z}/2).$$

So in order to determine $\operatorname{Tor}_n^{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2)$, it is sufficient to consider the long exact sequence

$$F^{\mathbb{Z}/2} \otimes \mathbb{Z}/2 = \ldots \longrightarrow \mathbb{Z}/4 \otimes \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \otimes \mathbb{Z}/2 \longrightarrow \ldots \longrightarrow \mathbb{Z}/4 \otimes \mathbb{Z}/2 \longrightarrow 0.$$

In the sequence above, the homomorphisms are given by $(\cdot 2) \otimes \mathrm{id}_{\mathbb{Z}/2}$. Notice that by Problem Sheet 1, Problem 1, it holds that $\mathbb{Z}/4 \otimes \mathbb{Z}/2 \cong \mathbb{Z}/\gcd(2,4) \cong \mathbb{Z}/2$ and the homomorphism is precisely the zero map. Consequently, $\mathrm{H}_n(F^{\mathbb{Z}/2} \otimes \mathbb{Z}/2) \cong \ker(\cdot 0) \,/\, \operatorname{im}(\cdot 0) \cong (\mathbb{Z}/2)/0 \cong \mathbb{Z}/2$ for all $n \geq 0$.
By the above, this yields

$$\operatorname{Tor}_n^{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2,$$

for all $n \geq 0$.

Now let's calculate $\operatorname{Ext}^n_{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2)$. By definition, it holds that

$$\operatorname{Ext}^n_{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathrm{H}^n(\operatorname{Hom}(F^{\mathbb{Z}/2}, \mathbb{Z}/2)),$$

where

$$\operatorname{Hom}(F^{\mathbb{Z}/2}, \mathbb{Z}/2) = \ldots \longleftarrow \operatorname{Hom}(\mathbb{Z}/4, \mathbb{Z}/2) \longleftarrow \ldots \longleftarrow \operatorname{Hom}(\mathbb{Z}/4, \mathbb{Z}/2) \longleftarrow 0.$$

Notice that $\operatorname{Hom}(\mathbb{Z}/4, \mathbb{Z}/2) \cong \mathbb{Z}/2$ since to define a homomorphism from $\mathbb{Z}/4$ to $\mathbb{Z}/2$, a generator of $\mathbb{Z}/4$ can either be mapped to $0 + \mathbb{Z}/2$ or to $1 + \mathbb{Z}/2$. The homomorphisms in the long exact sequence are given by the dual of multiplication by 2, which is the zero map in the depicted case. Hence, we obtain

$\mathrm{H}^n(\mathrm{Hom}(F^{\mathbb{Z}/2}, \mathbb{Z}/2)) \cong \ker(\cdot 0)/\operatorname{im}(\cdot 0) \cong (\mathbb{Z}/2)/0 \cong \mathbb{Z}/2$ for all $n \geq 0$.
This implies that

$$\mathrm{Ext}^n_{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2,$$

for all $n \geq 0$.

PROBLEM 6

Maria Morariu

a). Let p denote the given covering of S^1. We start by showing that $\tilde{\sigma}(1) - \tilde{\sigma}(0)$ does not depend on the choice of the lift $\tilde{\sigma}$. Let $\bar{\sigma}\colon [0,1] \to \mathbb{R}$ be a further lift of σ. Define $\bar{\sigma}'\colon [0,1] \to \mathbb{R}, \bar{\sigma}'(t) = \tilde{\sigma}(t) + \bar{\sigma}(0) - \tilde{\sigma}(0)$. This map is continuous with $\bar{\sigma}'(0) = \bar{\sigma}(0)$ and $e^{2\pi i \bar{\sigma}'(t)} = e^{2\pi i \tilde{\sigma}(t)} e^{2\pi i \bar{\sigma}(0)} \left(e^{2\pi i \tilde{\sigma}(0)}\right)^{-1} = \sigma(t)\sigma(0)\sigma(0)^{-1} = \sigma(t)$, so $\bar{\sigma}'$ is a lift of σ with $\bar{\sigma}'(0) = \bar{\sigma}(0)$. By the uniqueness in the lifting property of covers, it follows $\bar{\sigma}' = \bar{\sigma}$ and in particular $\bar{\sigma}(1) = \bar{\sigma}'(1) = \tilde{\sigma}(1) + \bar{\sigma}(0) - \tilde{\sigma}(0)$ and thus $\bar{\sigma}(1) - \bar{\sigma}(0) = \tilde{\sigma}(1) - \tilde{\sigma}(0)$. Therefore, $\tilde{\sigma}(1) - \tilde{\sigma}(0)$ does not depend on the choice of the lift $\tilde{\sigma}$ and we can define the map $\phi\colon C_1(S^1) \to \mathbb{R}$ as the linear map with $\phi(\sigma) = \tilde{\sigma}(1) - \tilde{\sigma}(0)$ for any 1-simplex σ. By defintion, this is a 1-cochain of S^1 with coefficients in $\mathbb{R}$.
Let us show that ϕ is actually a 1-cocycle. By Remark 5 in the lecture, this is the same as showing that ϕ is 0 on 1-boundaries. Let $\sigma\colon \Delta^2 \to S^1$ be a singular 2-simplex. We show that $\phi(d\sigma) = 0$. By definition $d\sigma = \sigma|_{[1,2]} - \sigma|_{[0,2]} + \sigma|_{[0,1]}$. Since Δ^2 is simply connected, the lifting property of covers implies that there exists a lift $\tilde{\sigma}\colon \Delta^2 \to \mathbb{R}$ such that $p \circ \tilde{\sigma} = \sigma$. Then $\tilde{\sigma}|_{[1,2]}, \tilde{\sigma}|_{[0,2]}, \tilde{\sigma}|_{[0,1]}$ are lifts of $\sigma|_{[1,2]}, \sigma|_{[0,2]}, \sigma|_{[0,1]}$. Hence, we have

$$\begin{aligned}
\phi(d\sigma) &= \phi(\sigma|_{[1,2]}) - \phi(\sigma|_{[0,2]}) + \phi(\sigma|_{[0,1]}) \\
&= \tilde{\sigma}|_{[1,2]}(1) - \tilde{\sigma}|_{[1,2]}(0) - \big(\tilde{\sigma}|_{[0,2]}(1) - \tilde{\sigma}|_{[0,2]}(0)\big) + \tilde{\sigma}|_{[0,1]}(1) - \tilde{\sigma}|_{[0,1]}(0) \\
&= \tilde{\sigma}(2) - \tilde{\sigma}(1) - \tilde{\sigma}(2) + \tilde{\sigma}(0) + \tilde{\sigma}(1) - \tilde{\sigma}(0) = 0.
\end{aligned}$$

Since all 1-boundaries can be written as finite sums of such $d\sigma$, it follows that ϕ is zero on 1-boundaries, so ϕ is a 1-cocycle.
Lastly, we show that ϕ generates $\mathrm{H}^1(S^1, \mathbb{R})$. Note that $\mathrm{H}_0(S^1) \cong \mathbb{Z}$ and it is in particular free, so $\mathrm{Ext}(\mathrm{H}_0(S^1), \mathbb{R}) \cong 0$ and by the universal coefficient theorem for cohomology, it follows that evaluation map $ev\colon \mathrm{H}^1(X; \mathbb{R}) \to \mathrm{Hom}(\mathrm{H}_1(S^1), \mathbb{R}), [\psi] \mapsto \psi$ is an isomorphism. Also, $\mathrm{H}_1(S^1) \cong \mathbb{Z}$, so we have a natural isomorphism $\mathrm{Hom}(\mathrm{H}_1(S^1), \mathbb{R}) \to \mathbb{R}, \psi \mapsto \psi([\sigma])$, where $\sigma\colon [0,1] \to S^1, t \mapsto e^{2\pi i t}$. Let us note that $\tilde{\sigma}\colon [0,1] \to \mathbb{R}, t \mapsto t$ is a lift of σ, so $\phi(\sigma) = \tilde{\sigma}(1) - \tilde{\sigma}(0) = 1 - 0 = 1$, which is a generator for $\mathbb{R}$, so $[\phi]$ is a generator for $\mathrm{H}^1(S^1, \mathbb{R})$.

b). Again, we start by showing that for any path $\sigma\colon [0,1] \to S^1$ and any lift $\tilde{\sigma}$ of σ, the expression $\lfloor \tilde{\sigma}(1) \rfloor - \lfloor \tilde{\sigma}(0) \rfloor$ does not depend on the choice of the lift. Let $\tilde{\sigma}$ and $\bar{\sigma}$ be two lifts of σ. In a), we have seen that
$\bar{\sigma}(1) - \bar{\sigma}(0) = \tilde{\sigma}(1) - \tilde{\sigma}(0)$ and thus $\tilde{\sigma}(1) - \bar{\sigma}(1) = \tilde{\sigma}(0) - \bar{\sigma}(0)$. For any $t \in [0,1]$, we have $e^{2\pi i \tilde{\sigma}(t)} = \sigma(t) e^{2\pi i \bar{\sigma}(t)}$, so $\tilde{\sigma} - \bar{\sigma} \in \mathbb{Z}$ and therefore $\{\tilde{\sigma}(t)\} = \{\bar{\sigma}(t)\}$, where

by $\{x\} = x - \lfloor x \rfloor$ we denote the fractional part of $x \in \mathbb{R}$. Hence, $\tilde{\sigma}(t) - \bar{\sigma}(t) = \lfloor \tilde{\sigma}(t) \rfloor + \{\tilde{\sigma}(t)\} - \lfloor \bar{\sigma}(t) \rfloor - \{\bar{\sigma}(t)\} = \lfloor \tilde{\sigma}(t) \rfloor - \lfloor \bar{\sigma}(t) \rfloor$. In particular,

$$\lfloor \tilde{\sigma}(1) \rfloor - \lfloor \bar{\sigma}(1) \rfloor = \tilde{\sigma}(1) - \bar{\sigma}(1) = \tilde{\sigma}(0) - \bar{\sigma}(0) = \lfloor \tilde{\sigma}(0) \rfloor - \lfloor \bar{\sigma}(0) \rfloor.$$

Thus, we can define the map $\phi\colon C_1(S^1) \to \mathbb{Z}$ by linearly extending $\phi(\sigma) = \lfloor \tilde{\sigma}(1) \rfloor - \lfloor \tilde{\sigma}(0) \rfloor$.
The fact that ϕ is a cocycle follows exactly as in a), since the floors of two equal numbers are equal. Also, as in a), we get a natural isomorphism $H^1(S^1, \mathbb{Z}) \to \mathbb{Z}, [\psi] \mapsto \psi(\sigma)$, where $\sigma\colon [0,1] \to S^1, t \mapsto e^{2\pi i t}$. Observe that $[\phi]$ is mapped to 1 under this isomorphism, which is a generator for $\mathbb{Z}$ and thus $[\phi]$ is a generator for $\mathrm{H}^1(S^1, \mathbb{Z})$.

PROBLEM 7

Clara Bonvin

The UCT for cohomology gives the following SES :

$$0 \longrightarrow \mathrm{Ext}(H_0(X,\mathbb{Z}),\mathbb{Z}) \longrightarrow H^1(X;\mathbb{Z}) \longrightarrow \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z}) \longrightarrow 0$$

Note that $H_0(X,\mathbb{Z})$ is a free $\mathbb{Z}$ Module, therefore we have $\mathrm{Ext}(H_0(X,\mathbb{Z}),\mathbb{Z}) = 0$.
From the above SES, we get : $H^1(X,\mathbb{Z}) \cong \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z})$.
Therefore, it suffices to show that $\mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z})$ is torsion free to deduce that $H^1(X,\mathbb{Z})$ is torsion free as well.
To show that $\mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z})$ is torsion free, we show that $\mathrm{Hom}(M,\mathbb{Z})$ is torsion free for any $\mathbb{Z}$ Module M. We consider its torsion group $\mathrm{T}(\mathrm{Hom}(M,\mathbb{Z})) = \{\varphi \in \mathrm{Hom}(M,\mathbb{Z}) : \exists \lambda \in \mathbb{Z} \setminus \{0\} \text{ with } \lambda\varphi = 0\}$ and show that it is trivial.
Let $\varphi \in \mathrm{T}(\mathrm{Hom}(M,\mathbb{Z}))$, then $\forall m \in M$, there exists some $\lambda \neq 0$ such that $\lambda\varphi(m) = 0$.
This gives $\forall m \in M : \varphi(m) = 0 \Rightarrow \varphi = 0$ and therefore we get $\mathrm{T}(\mathrm{Hom}(M,\mathbb{Z})) = 0$.

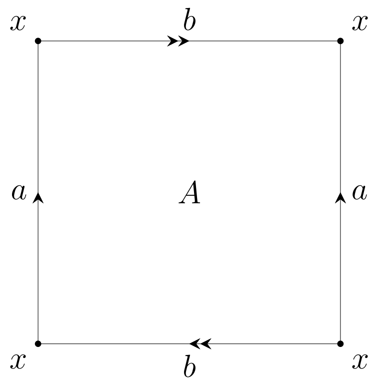
Problem 8

Sina Keller and Tristan Lovsin

Since b) and c) are more intuitive to understand we put subpart a) at the end.

b).

Cellular homology. We look at the Klein bottle K^2 as the square with the edges identified as shown in the following picture:



We then have the following cochains in cohomology:

$$\begin{aligned} 0-\text{cochains:} \quad & \varphi : x \mapsto 1 \\ 1-\text{cochains:} \quad & \alpha : \begin{cases} a & \mapsto 1 \\ b & \mapsto 0 \end{cases} \text{ and } \beta : \begin{cases} a & \mapsto 0 \\ b & \mapsto 1 \end{cases} \\ 2-\text{cochains:} \quad & \gamma : A \mapsto 1 \end{aligned}$$

These are maps from the cellular chain group to $\mathbb{Z}$ or $\mathbb{F}_2$ respectively and are generating $C^i(T^2; \mathbb{Z} \text{ or } \mathbb{F}_2)$ for $i = 0, 1, 2$.
Now let's look at the boundary maps.

$$\begin{aligned} d^0 : C^0 &\to C^1 \\ \varphi &\mapsto d^0(\varphi) \\ d^0(\varphi)(a) &= \varphi(d_1(a)) = \varphi(x - x) = 0 \\ d^0(\varphi)(b) &= \varphi(d_1(b)) = \varphi(x - x) = 0 \end{aligned}$$

From this we get the kernel and image of d^0:

$$\ker(d^0) = \langle \varphi \rangle \tag{1}$$

$$\mathrm{im}(d^0) = 0 \tag{2}$$

Now we do the same for d^1:

$$\begin{aligned} d^1 : C^1 &\to C^2 \\ \alpha &\mapsto d^1(\alpha) \\ d^1(\alpha)(A) &= \alpha(d_2(A)) = \alpha(b) - \alpha(a) + \alpha(b) + \alpha(a) = 0 - 1 + 0 + 1 = 0 \\ d^1(\beta)(A) &= \beta(d_2(A)) = \beta(b) - \beta(a) + \beta(b) + \beta(a) = 2\beta(b) = 2 \implies d^1(\beta) = 2\gamma \end{aligned}$$

Here we get two different cases for coefficients in $\mathbb{Z}$ and $\mathbb{F}_2$.

	$\mathbb{Z}$	$\mathbb{F}_2$
(3)	$\ker(d^1) = \langle \alpha \rangle$	$\ker(d^1) = \langle \alpha, \beta \rangle$
(4)	$\mathrm{im}(d^1) = \langle 2\gamma \rangle$	$\mathrm{im}(d^1) = 0$

Now let's do the same for d^2:

$$\begin{aligned} d^2 : C^2 &\to C^3 \\ \gamma &\mapsto d^2(\gamma) \\ d^2(\gamma)(0) &= \gamma(d_3(0)) = 0 \end{aligned}$$

We get for both $\mathbb{Z}$ and $\mathbb{F}_2$ that:

$$\begin{aligned} \ker(d^2) &= \langle \gamma \rangle \\ \mathrm{im}(d^2) &= 0 \end{aligned} \tag{5}$$

Now we combine all these equations to get the i-th cohomology with coefficient in $\mathbb{Z}$ and $\mathbb{F}_2$:

	$\mathbb{Z}$	$\mathbb{F}_2$
$H^0 \cong \ker(d^0)/\mathrm{im}(d^{-1}) \overset{(1)}{\cong}$	$\mathbb{Z}/0 = \mathbb{Z}$	$\mathbb{F}_2/0 = \mathbb{F}_2$
$H^1 \cong \ker(d^1)/\mathrm{im}(d^0) \overset{(2),(3)}{\cong}$	$\mathbb{Z}/0 = \mathbb{Z}$	$\mathbb{F}_2^2/0 = \mathbb{F}_2^2$
$H^2 \cong \ker(d^2)/\mathrm{im}(d^1) \overset{(4),(5)}{\cong}$	$\mathbb{Z}/2\mathbb{Z} = \mathbb{F}_2$	$\mathbb{F}_2/0 = \mathbb{F}_2$

Checking with UCT. Now let's check if this result is the same as in the UCT of cohomology. We check the following SES, where A is either $\mathbb{Z}$ or $\mathbb{F}_2$ and C our chain complex of K^2:

$$0 \to \mathrm{Ext}(H_{i-1}(C), A) \to H^i(C; A) \to \mathrm{Hom}(H_i(C), A) \to 0$$

We have to check the following SES:

$$0 \to \mathrm{Ext}(H_{-1}(K^2), \mathbb{Z}) \to H^0(K^2; \mathbb{Z}) \to \mathrm{Hom}(H_0(K^2), \mathbb{Z}) \to 0 \tag{6}$$
$$0 \to \mathrm{Ext}(H_0(K^2), \mathbb{Z}) \to H^1(K^2; \mathbb{Z}) \to \mathrm{Hom}(H_1(K^2), \mathbb{Z}) \to 0 \tag{7}$$
$$0 \to \mathrm{Ext}(H_1(K^2), \mathbb{Z}) \to H^2(K^2; \mathbb{Z}) \to \mathrm{Hom}(H_2(K^2), \mathbb{Z}) \to 0 \tag{8}$$
$$0 \to \mathrm{Ext}(H_{-1}(K^2), \mathbb{F}_2) \to H^0(K^2; \mathbb{F}_2) \to \mathrm{Hom}(H_0(K^2), \mathbb{F}_2) \to 0 \tag{9}$$
$$0 \to \mathrm{Ext}(H_0(K^2), \mathbb{F}_2) \to H^1(K^2; \mathbb{F}_2) \to \mathrm{Hom}(H_1(K^2), \mathbb{F}_2) \to 0 \tag{10}$$
$$0 \to \mathrm{Ext}(H_1(K^2), \mathbb{F}_2) \to H^2(K^2; \mathbb{F}_2) \to \mathrm{Hom}(H_2(K^2), \mathbb{F}_2) \to 0 \tag{11}$$

From AlgTopo I we remember the homology groups of K^2:

$$H_i \cong \begin{cases} \mathbb{Z} & \text{i=0} \\ \mathbb{Z} \oplus \mathbb{F}_2 & \text{i=1} \\ 0 & \text{else} \end{cases} \tag{12}$$

We know from Prop 8, that if A is free $\mathrm{Ext}(A, B) \cong 0$.

0th cohomology. Therefore (6) and (9) with help of (12) becomes the following as desired:

$$\begin{aligned} (6): 0 \to & H^0(K^2; \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \to 0 \\ \Rightarrow & H^0(K^2; \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \end{aligned} \tag{13}$$
$$\begin{aligned} (9): 0 \to & H^0(K^2; \mathbb{F}_2) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{F}_2) \to 0 \\ \Rightarrow & H^0(K^2; \mathbb{F}_2) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{F}_2) \cong \mathbb{F}_2 \end{aligned} \tag{14}$$

1st cohomology. Similarly (7) and (10) with (12) and the fact that $\mathbb{Z}$ is free turns into the following, as desired:

$$\begin{aligned} (7): 0 \to & H^1(K^2; \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z} \oplus \mathbb{F}_2, \mathbb{Z}) \to 0 \\ \Rightarrow & H^1(K^2; \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z} \oplus \mathbb{F}_2, \mathbb{Z}) \cong \mathbb{Z} \\ (10): 0 \to & H^1(K^2; \mathbb{F}_2) \to \mathrm{Hom}(\mathbb{Z} \oplus \mathbb{F}_2, \mathbb{F}_2) \to 0 \\ \Rightarrow & H^1(K^2; \mathbb{F}_2) \cong \mathrm{Hom}(\mathbb{Z} \oplus \mathbb{F}_2, \mathbb{F}_2) \overset{*}{\cong} \mathbb{F}_2^2 \end{aligned}$$

* is deduced from the fact that $\mathrm{Hom}(A \oplus B, C) \cong \mathrm{Hom}(A, C) \oplus \mathrm{Hom}(B, C)$ which in this case yields

$$\mathrm{Hom}(\mathbb{Z} \oplus \mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{F}_2) \oplus \mathrm{Hom}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$$

2nd cohomology. Now we need the fact that $\mathrm{Ext}(A \oplus B, C) \cong \mathrm{Ext}(A, C) \oplus \mathrm{Ext}(B, C)$ from Prop 8. This turns (8) and (11) with help of (12) into:

$$(8) : 0 \to \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{F}_2, \mathbb{Z}) \to H^2(K^2; \mathbb{Z}) \to 0$$

$$\mathrm{Ext}(\mathbb{F}_2, \mathbb{Z}) \cong 0 \oplus \mathrm{Ext}(\mathbb{F}_2, \mathbb{Z}) \cong \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{F}_2, \mathbb{Z}) \cong H^2(K^2; \mathbb{Z}) \tag{15}$$

$$(11) : 0 \to \mathrm{Ext}(\mathbb{Z}, \mathbb{F}_2) \oplus \mathrm{Ext}(\mathbb{F}_2, \mathbb{F}_2) \to H^2(K^2; \mathbb{F}_2) \to 0$$

$$\mathrm{Ext}(\mathbb{F}_2, \mathbb{F}_2) \cong 0 \oplus \mathrm{Ext}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}(\mathbb{Z}, \mathbb{F}_2) \oplus \mathrm{Ext}(\mathbb{F}_2, \mathbb{F}_2) \cong H^2(K^2; \mathbb{F}_2) \tag{16}$$

Now we want to calculate Ext for (15) and (16). We remember the definition of Ext from the lecture:

$$\mathrm{Ext}(M, N) := H^1(\mathrm{Hom}(F^M, N))$$

Therefore we try to calculate the following:

$$\mathrm{Ext}(\mathbb{F}_2, \mathbb{Z}) = H^1(\mathrm{Hom}(F^{\mathbb{F}_2}, \mathbb{Z})) \tag{17}$$

Let $F^{\mathbb{F}_2} := 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0$ be a free resolution of $\mathbb{F}_2$. Then by definition $\mathrm{Hom}(F^{\mathbb{F}_2}, \mathbb{Z})$ form the corresponding cochain complexes and we calculate $\mathrm{im}(d^0)$ and $\ker(d^1)$. Let σ, τ be an arbitrary element in C^0, C^1 respectively. Let a, b be elements that were defined in part 1.

$$\begin{aligned}
d^0 : \qquad & C^0 \to C^1 \\
& \sigma \mapsto d^0(\sigma) \\
d^0(\sigma)(b) &= \sigma(d_1(b)) = \sigma(2b) = 2 \\
d^1 : \qquad & C^1 \to C^2 \\
& \tau \mapsto d^1(\tau) \\
d^1(\tau)(0) &= 0
\end{aligned}$$

Therefore we have that $\mathrm{im}(d^0) \cong 2\mathbb{Z}$ and $\ker(d^1) \cong \mathbb{Z}$. Now we have that

$$\begin{aligned}
H^1 &\cong {}^{\ker(d^1)}\!\big/_{\mathrm{im}(d^0)} \cong {}^{\mathbb{Z}}\!\big/_{2\mathbb{Z}} \cong \mathbb{F}_2 \\
&\overset{(17)}{\cong} \mathrm{Ext}(\mathbb{F}_2, \mathbb{Z})
\end{aligned} \tag{18}$$

This is exactly the result we expected from our calculations from cellular cohomology.
Now onto (16) where we try to calculate the following:

$$\mathrm{Ext}(\mathbb{F}_2, \mathbb{F}_2) = H^1(\mathrm{Hom}(F^{\mathbb{F}_2}, \mathbb{F}_2)) \tag{19}$$

We have that

$$\mathrm{Hom}(\mathbb{Z}, \mathbb{F}_2) \cong \mathbb{F}_2$$

and

$$\begin{aligned}
d^0 : \quad & C^0 \to C^1 \\
& \sigma \mapsto d^0(\sigma) \\
& d^0(\sigma)(b) = \sigma(d_1(b)) = \sigma(2b) = 2\sigma \equiv 0 \\
d^1 : \quad & C^1 \to C^2 \\
& \tau \mapsto d^1(\tau) \\
& d^1(\tau)(0) = 0
\end{aligned}$$

Therefore we have that $\mathrm{im}(d^0) \cong 2\mathbb{F}_2 \cong 0$ and $\ker(d^1) \cong \mathbb{F}_2$, thus

$$\begin{aligned}
H^1 &\cong {\ker(d^1)}\big/{\mathrm{im}(d^0)} \cong {\mathbb{F}_2}\big/{0} \cong \mathbb{F}_2 \\
&\overset{(19)}{\cong} \mathrm{Ext}(\mathbb{F}_2, \mathbb{F}_2) \qquad (20)
\end{aligned}$$

Therefore all our results align with UCT and we are done.

c).

Cellular cohomology. We know from AlgTopo I that $\mathbb{R}P^n$ has the following cellular homology:

$$\begin{aligned}
n-\text{cell}: \quad & c_n \\
n-1-\text{cell}: \quad & c_{n-1} \\
\vdots \quad & \vdots \\
0-\text{cell}: \quad & c_0
\end{aligned}$$

With the following boundary maps:

$$d_i(c_i) = \begin{cases} 0 & \text{i odd} \\ 2c_{i-1} & \text{i even} \end{cases} \qquad (21)$$

Therefore we construct the following cellular cohomology:

$$\begin{aligned}
n-\text{cell}: \quad & \varphi_n : c_n \mapsto 1 \\
n-1-\text{cell}: \quad & \varphi_{n-1} : c_{n-1} \mapsto 1 \\
\vdots \quad & \vdots \\
0-\text{cell}: \quad & \varphi_0 : c_0 \mapsto 1
\end{aligned}$$

Now we look at the boundary maps in cohomology:

$$d^i(\varphi_i)(c_{i+1}) = \varphi_i(d_{i+1}(c_{i+1})) \overset{(21)}{=} \begin{cases} 2\varphi_i & \text{i odd} \\ 0 & \text{i even} \end{cases}$$

This gives us the kernel and the image:

$$\mathbb{Z} \qquad\qquad\qquad\qquad\qquad\qquad \mathbb{F}_2$$

$$\ker(d^i) = \begin{cases} 0 & \text{i odd} \\ \langle\varphi_i\rangle & \text{i even} \end{cases} \qquad \ker(d^i) = \begin{cases} \langle\varphi_i\rangle & \text{i odd} \\ \langle\varphi_i\rangle & \text{i even} \end{cases} \tag{22}$$

$$\operatorname{im}(d^i) = \begin{cases} \langle 2\varphi_i\rangle & \text{i odd} \\ 0 & \text{i even} \end{cases} \qquad \operatorname{im}(d^i) = \begin{cases} 0 & \text{i odd} \\ 0 & \text{i even} \end{cases} \tag{23}$$

Now we can put this together to get the cohomology groups with coefficient in A which is $\mathbb{Z}$ or $\mathbb{F}_2$ respectively:

$$H^i(K^2; A) \cong {\ker(d^i)}/{\operatorname{im}(d^{i-1})} \overset{(22),(23)}{\cong} \begin{cases} \mathbb{Z}: & \mathbb{F}_2: & \\ \mathbb{Z}/0 \cong \mathbb{Z} & \mathbb{F}_2/0 \cong \mathbb{F}_2 & i = n \text{ with n odd or } i = 0 \\ 0/0 \cong 0 & \mathbb{F}_2/0 \cong \mathbb{F}_2 & 0 < i < n, i \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} \cong \mathbb{F}_2 & \mathbb{F}_2/0 \cong \mathbb{F}_2 & 0 < i \leq n, i \text{ even} \end{cases}$$

Checking with UCT. Now we have to check if those results are in alignment with UCT. Again we have that A is either $\mathbb{Z}$ or $\mathbb{F}_2$ and C our chain complex of $\mathbb{R}P^n$

$$0 \to \operatorname{Ext}(H_{i-1}(C), A) \to H^i(C; A) \to \operatorname{Hom}(H_i(C), A) \to 0$$

We remember the homology group of $\mathbb{R}P^2$ from AlgTopo I:

$$H_i(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & \text{i=0, i=n and n odd} \\ \mathbb{F}_2 & \text{i odd and } 0 < i < n \\ 0 & \text{else} \end{cases} \tag{24}$$

We get now the following different cases for the SES with help of Prop 8:

$$0 \to H^i(\mathbb{R}P^n; A) \to \operatorname{Hom}(\mathbb{Z}, A) \to 0 \quad i = 0, i = n \text{ and } n \text{ odd} \tag{25}$$

$$0 \to H^i(\mathbb{R}P^n; A) \to \operatorname{Hom}(\mathbb{F}_2, A) \to 0 \quad i \text{ odd and } 0 < i < n \tag{26}$$

$$0 \to \operatorname{Ext}(\mathbb{F}_n, A) \to H^i(\mathbb{R}P^2; A) \to 0 \quad i \text{ even and } 0 < i < n \tag{27}$$

We have already calculated in (13) and (14) what $\operatorname{Hom}(\mathbb{Z}, A)$ is for both cases. Therefore (25) aligns with UCT.
We have calculated $\operatorname{Ext}(\mathbb{F}_n, A)$ in (18) and (20) for both cases. Therefore (27) aligns with UCT as well.
Now let's look at both cases for (26). In the case of $A = \mathbb{Z}$ we have that $\operatorname{Hom}(\mathbb{F}_2, \mathbb{Z}) \cong 0$ and when $A = \mathbb{F}_2$ we have $\operatorname{Hom}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2$, yielding the desired result for all cases and we are done.

a).

Cellular homology. We view $T^n = {}^{I^n}\!/\!_{\sim}$ as the n-dim cube with the equivalence relation $\sim$ which identifies opposite facets of the boundary.
We construct our cells similarly to a hypercube without identifying opposite facets. As Wikipedia stated, we imagine the hypercube in a Cartesian coordinate system. Then there exists for every i-dim cell i coordinate axes that are parallel to this element. This results in $\binom{n}{k}$ elements. Differently to Wikipedia, we don't have to multiply by 2^{n-i} for the total amount of cells, because the parallel cells on the other side of the hypercube are identified. [1]
So we have $\binom{n}{i}$ i-cells for each i-dim cochain group C_i. We call them c^i_k for $i = 0, \dots, n$ and $k = 0, \dots, \binom{n}{i}$. Now we compute the boundary maps d_i.
Quick reminder how cellular homology works. We look at embeddings f of the n-dimensional ball B^n and its boundary ∂B^n into the $K^{(n)}$ and $K^{(n-1)}$ skeletons respectively, where these maps are injective on the interior of B^n but not necessarily on the boundary. Then we took the quotient space ${}^{K^{(n)}}\!/\!_{K^{(n-1)}} \simeq \bigvee S^n$ and looked at the projection onto one of those spheres. This projection we called p[2]. In this exercise we will use the notation from AlgTopo I and the solution of exercise 3 of the exercise sheet 7.
For d_i we consider any of the maps $p_{c^{i-1}_k} f_{\partial c^i_m} : \partial I^i \to S^{i-1}$. We note that there are two opposite facets of I^i in whose interiors this map restricts to a homeomorphism. The map collapses the rest of ∂I^i to a point in S^{i-1}. The degree of $p_{c^{i-1}_k} f_{\partial c^i_m}$ is therefore the sum of the two local degrees at any two points in q_1, q_2 in the two first-mentioned facets which get mapped to the same point in T^n. Now we note that the restriction of $p_{c^{i-1}_k} f_{\partial c^i_m}$ to these faces are obtained from one another by precomposing with an orientation-reversing map. Therefore the sum of these local degrees vanishes. Therefore we have that $\deg(d_i) = 0$ for all i. [3]
For the cellular cohomology we take the the cochains $C^i := \mathrm{Hom}(C_i, A)$, for A either $\mathbb{Z}$ or $\mathbb{F}_2$. We again have $\binom{n}{i}$ i-cochains for each i-dim cochain group C^i.
Now we want to calculate the coboundary maps d^i. We know that the coboundary maps are the transpose of the boundary maps and since the boundary maps are all 0, we have that $d^i = 0$ for all i as well.
We have that $\ker(d^i) \cong A^{\binom{n}{i}}$ and $\mathrm{im}(d^i) \cong 0$ for all i. We immediately get the cohomology groups:

$$H^i(T^n; \mathbb{Z}) \cong {}^{\ker(d^i)}\!/\!_{\mathrm{im}(d^{i-1})} \cong {}^{\mathbb{Z}^{\binom{n}{i}}}\!/\!_{0} \cong \mathbb{Z}^{\binom{n}{i}}$$

$$H^i(T^n; \mathbb{F}_2) \cong {}^{\ker(d^i)}\!/\!_{\mathrm{im}(d^{i-1})} \cong {}^{\mathbb{F}_2^{\binom{n}{i}}}\!/\!_{0} \cong \mathbb{F}_2^{\binom{n}{i}}$$

[1] Wikipedia: Hyperwüfel
[2] A detailed explanation of this is found in the lecture notes of Lecture 25 AlgTopo I starting at page 6.
[3] Generalized version of proof for exercise 3 on exercise sheet 7 in AlgTopo I

Checking with UCT. Now we check if this result aligns with UCT for cohomology:

$$0 \to \mathrm{Ext}(H_{i-1}(C), A) \to H^i(C; A) \to \mathrm{Hom}(H_i(C), A) \to 0 \tag{28}$$

We know that $H_i(T^n) = \mathbb{Z}^{\binom{n}{i}}$ from AlgTopo I. This is just a finite number of copies of $\mathbb{Z}$ and therefore free. We know from Prop 8, that if A is free $\mathrm{Ext}(A, B) \cong 0$. We note that $\mathrm{Hom}(\mathbb{Z}^i, \mathbb{Z}) \cong \mathbb{Z}^i$ and $\mathrm{Hom}(\mathbb{Z}^i, \mathbb{F}_2) \cong \mathbb{F}_2^i$. Therefore (28) in our case becomes:

$$\begin{aligned} 0 \to &H^i(T^n; A) \to \mathrm{Hom}(\mathbb{Z}^{\binom{n}{i}}, A) \to 0 \\ \Rightarrow 0 \to &H^i(T^n; A) \to A^{\binom{n}{i}} \to 0 \\ \Rightarrow &H^i(T^n; \mathbb{Z}) \cong \mathbb{Z}^{\binom{n}{i}} \text{ and} \\ &H^i(T^n; \mathbb{F}_2) \cong \mathbb{F}_2^{\binom{n}{i}} \end{aligned}$$

This is exactly what we expect from cellular cohomology and we are done.

PROBLEM 9

no solutions for starred problems