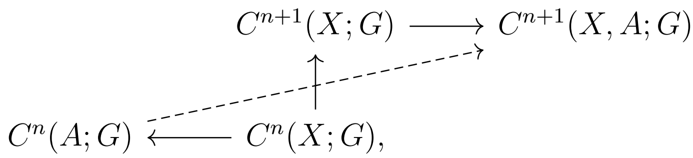
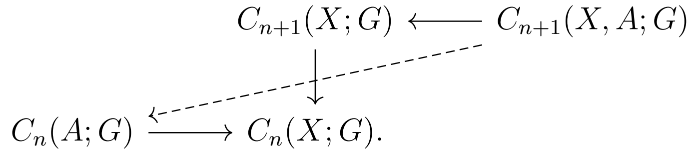
Problem 1

The duality relationship between the connecting homomorphisms $\delta\colon H^n(A;G) \to H^{n+1}(X,A;G)$ and $\partial\colon H_{n+1}(X,A) \to H_n(A)$ is given by the following commutative diagram:

$$\begin{array}{ccc} H^n(A;G) & \xrightarrow{\quad\delta\quad} & H^{n+1}(X,A;G) \\ {\scriptstyle ev}\downarrow & & \downarrow{\scriptstyle ev} \\ \mathrm{Hom}_{\mathbb{Z}}(H_n(A),G) & \xrightarrow{\partial^*} & \mathrm{Hom}_{\mathbb{Z}}(H_{n+1}(X,A),G). \end{array}$$

To verify commutativity, recall how the two connecting homomorphisms are defined, via the diagrams





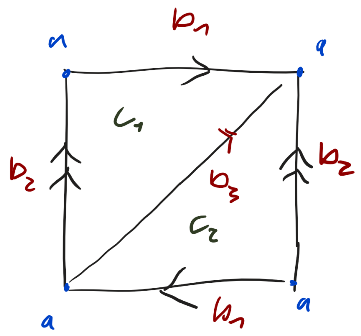
The connecting homomorphisms are represented by the dashed arrows, which are well-defined only when the cochain and chain groups are replaced by cohomology and homology groups, respectively.

To show that $\mathrm{ev}\delta = \partial^*\mathrm{ev}$, start with an element $\alpha \in H^n(A;G)$ represented by a cocycle $\varphi \in C^n(A;G)$. To compute $\delta(\alpha)$ we first extend φ to a cochain $\varphi \in C^n(X;G)$, say by letting it take the value 0 on singular simplices not in A. Then we compose φ with $\partial\colon C_{n+1}(X) \to C_n(X)$ to get a cochain $\varphi\partial \in C^{n+1}(X;G)$, which actually lies in $C^{n+1}(X,A;G)$ since the original φ was a cocycle in A. This cochain $\varphi\partial \in C^{n+1}(X,A;G)$ represents $\delta(\alpha)$ in $H^{n+1}(X,A;G)$. Now we apply the map ev, which simply restricts the domain of $\varphi\partial$ to relative cycles in $C_{n+1}(X,A)$ that is, $(n+1)$-chains in X whose boundary lies in A. On such chains we have $\varphi\partial = \alpha\partial$ since the extension of α to φ is irrelevant. The net result of all this is that $\mathrm{ev}\delta(\alpha)$ is represented by $\alpha\partial$. Let us compare this with $\partial^*\mathrm{ev}(\alpha)$. Applying ev to φ restricts its domain to cycles in A. Then applying ∂^* composes with the map which sends a relative $(n+1)$-cycle in X to its boundary in A. Thus $\partial^*\mathrm{ev}(\alpha)$ is represented by $\alpha\partial$ just as $\mathrm{ev}\delta(\alpha)$ was, and so the square commutes.

Problem 2

Noah Stäuble & Philip Sandt & Richard von Moos

Co-homology of the Klein Bottle K with coefficients in $\mathbb{Z}$, $H_\bullet(K;\mathbb{Z})$. Consider the following delta-complex structure of K:



Resulting in the chain complex:

$$0 \to \langle c_1, c_2 \rangle \xrightarrow{d_2} \langle b_1, b_2, b_3 \rangle \xrightarrow{d_1} \langle a \rangle \to 0.$$

Where the boundary operators behave in the following way:

$$\begin{cases} d_1(b_i) = a - a = 0 \; \forall i = 1,2,3 \\ d_2(c_1) = b_1 + b_2 - b_3 \qquad\qquad d_2(c_2) = b_1 - b_2 + b_3 \end{cases} \tag{1}$$

From this we can immediately deduce all interesting subgroups, except the kernel of d_2. To this end, consider $\sigma = \alpha c_1 + \beta c_2 \in C_2^\Delta(K;\mathbb{Z})$ and observe

$$d(\sigma) = (\alpha - \beta)b_1 + (\alpha + \beta)b_2 - (\alpha + \beta)b_3 = 0 \Leftrightarrow \alpha = \beta = 0$$

Therefore, $\ker(d_2) = 0$ This yields the homology groups of the delta-complex $H_n^\Delta(K;\mathbb{Z})$ for the Klein Bottle with $\mathbb{Z}$-coefficients:

$$\begin{cases} H_0^\Delta(K;\mathbb{Z}) = \langle a \rangle \\ H_1^\Delta(K;\mathbb{Z}) = \langle b_1, b_2, b_3 \rangle / \langle b_1 + b_2 - b_3, b_1 - b_2 + b_3 \rangle \cong \langle b_1, b_2 \rangle / \langle 2b_2 \rangle \\ H_2^\Delta(K;\mathbb{Z}) = 0 \end{cases} \tag{2}$$

In degrees $n = 0, 1$, the respective preceding groups of lower degrees are free. Therefore, UCT states that $\mathrm{ev}^{-1} \colon \mathrm{Hom}_\mathbb{Z}(H_n(K;\mathbb{Z}), \mathbb{Z}) \to H^n(K;\mathbb{Z})$ is an isomorphism and yields a basis of co-homology. Denote by $\phi_a \colon a \mapsto 1 \in \mathrm{Hom}_\mathbb{Z}(H_0(K;\mathbb{Z}), \mathbb{Z})$, $\phi_{b_1} \colon b_1 \mapsto 1, b_2 \mapsto 0 \in \mathrm{Hom}_\mathbb{Z}(H_1(K;\mathbb{Z}), \mathbb{Z})$.
For degree $n = 2$ we can consider the dual basis of the simplicial cochain complex $C^2_\Delta(K;\mathbb{Z}) = \langle \phi_{c_1}, \phi_{c_2} \rangle$, where $\phi_i(c_j) = \delta_{ij}$, for $i, j = 1, 2$ and manually calculate the quotient of cocycles and coboundaries.

$$Z^2_\Delta(K;\mathbb{Z}) = C^2_\Delta(K;\mathbb{Z})$$

$$\begin{aligned} B^2_\Delta(K;\mathbb{Z}) = \mathrm{Im}(d^2) &= \langle \phi_{b_i} \circ d_2 : i = 1,2,3 \rangle \\ &= \langle \phi_{c_1} + \phi_{c_2}, \phi_{c_1} - \phi_{c_2}, -\phi_{c_1} + \phi_{c_2} \rangle \end{aligned}$$

This allows us to express the cohomology groups of K with $\mathbb{Z}$-coefficients

$$(3) \quad \begin{cases} H^0_\Delta(K;\mathbb{Z}) = \langle \phi_a \rangle \cong \mathbb{Z} \\ H^1_\Delta(K;\mathbb{Z}) = \langle \phi_{b_1} \rangle \cong \mathbb{Z} \\ H^2_\Delta(K;\mathbb{Z}) = \langle \phi_{c_1}, \phi_{c_2} \rangle / \langle \phi_{c_1} + \phi_{c_2}, \phi_{c_1} - \phi_{c_2} \rangle \cong \langle \phi_{c_1} \rangle / \langle 2\phi_{c_1} \rangle \cong \mathbb{Z}/2 \end{cases}$$

To discover the ring structure of $H^\bullet(K;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$ it remains to calculate the behavior of the cup product. From the degree formula it follows that only $\phi_{b_1} \smile \phi_{b_1}$ can be non-zero, aside from products with the unit element ϕ_a.
To this end, let again $\sigma = \alpha c_1 + \beta c_2 \in C_2^\Delta(K;\mathbb{Z})$ and denote by $e_0, e_1, e_2, e_3 \in K$ the vertices of $c_1 = [e_0, e_1, e_2]$ and $c_2 = [e_3, e_0, e_2]$

$$\begin{aligned} \phi_{b_1} \smile \phi_{b_1}(\sigma) &= \alpha \phi_{b_1}(c_1|_{[e_0,e_1]}) \phi_{b_1}(c_1|_{[e_1,e_2]}) + \beta \phi_{b_1}(c_2|_{[e_3,e_0]}) \phi_{b_1}(c_2|_{[e_0,e_2]}) \\ &= \alpha \phi_{b_1}(b_2) \phi_{b_1}(b_1) + \beta \phi_{b_1}(b_1) \phi_{b_1}(b_3) = 0 \end{aligned}$$

We found that all products except with ϕ_a vanish, resulting in $H^\bullet(K;\mathbb{Z}) \cong \mathbb{Z}[X_1, X_2]/(X_i X_j, 2X_2)$ with $\deg(X_i) = i$.

Co-homology of the Klein Bottle K with coefficients in $\mathbb{F}_2$, $H_\bullet(K;\mathbb{F}_2)$. Analogously to $\mathbb{Z}$-coefficients UCT yields a dual basis in degrees $n = 0, 1$

$$(4) \quad \begin{cases} H^0(K;\mathbb{F}_2) = \langle \phi_a \rangle \\ H^1(K;\mathbb{F}_2) = \langle \phi_{b_1}, \phi_{b_2} \rangle \end{cases}$$

To get a basis of $H^2(K;\mathbb{F}_2)$ we again work with the quotient of cocycles and coboundaries. Note that

$$Z^2_\Delta(K;\mathbb{F}_2) = C^2_\Delta(K;\mathbb{F}_2)$$

$$B^2_\Delta(K;\mathbb{F}_2) = \mathrm{Im}(d^2) = \langle \phi_{c_1} + \phi_{c_2} \rangle$$

Thus we can express $H^2_\Delta(K;\mathbb{F}_2)$ as

$$(5) \quad \begin{cases} H^2(K;\mathbb{F}_2) = \langle \phi_{c_1}, \phi_{c_2} \rangle / \langle \phi_{c_1} + \phi_{c_2} \rangle \cong \langle \phi_{c_1} \rangle \end{cases}$$

As for products in $H^\bullet(K;\mathbb{F}_2) \cong \mathbb{F}_2 \oplus (\mathbb{F}_2 \oplus \mathbb{F}_2) \oplus \mathbb{F}_2$, again because of degree, the only cochains in $H_1(K;\mathbb{F}_2)$ can have a non-zero product, aside from products with ϕ_a. Let $\sigma = \alpha c_1 + \beta c_2 \in C_2^\Delta(K;\mathbb{F}_2)$ and consider its value under a general cup product of $\phi_{b_1}, \phi_{b_2}, \phi_{b_3} \in C^2_\Delta(K;\mathbb{F}_2)$.

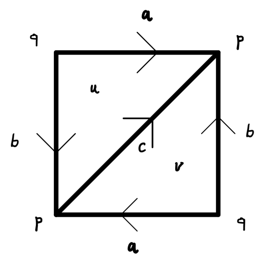
$$\begin{aligned} \phi_{b_i} \smile \phi_{b_j}(\sigma) &= \alpha \phi_{b_i}(c_1|_{[e_0,e_1]}) \phi_{b_j}(c_1|_{[e_1,e_2]}) + \beta \phi_{b_i}(c_2|_{[e_3,e_0]} \phi_{b_j}(c_2|_{[e_0,e_2]}) \\ &= \alpha \phi_{b_i}(b_2) \phi_{b_j}(b_1) + \beta \phi_{b_i}(b_1) \phi_{b_j}(b_3) \end{aligned}$$

which is only non-zero if

- $i = 2$, $j = 1$, which results in $\phi_{b_1} \smile \phi_{b_2} = \phi_{c_1}$.
- $i = 1$, $j = 3$, which descends in homology to the relation $\phi_{b_2} \smile \phi_{b_1} = \phi_{c_1}$. By which we satisfy the graded commutativity.

This allows, us to write the homology ring with $\mathbb{F}_2$-Coefficients as $H^\bullet(K;\mathbb{F}_2) \cong \mathbb{F}_2[X, Y]/\langle X^2, Y^2 \rangle$, where the degree of both X, Y is $\deg(X) = \deg(Y) = 1$.

Co-homology of $X = \mathbb{R}P^2$ with coefficients in $\mathbb{Z}$, $H_\bullet(X;\mathbb{Z})$.



The non-vanishing part of the simplicial complex is

$$\mathbb{Z}u \oplus \mathbb{Z}v \to \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \to \mathbb{Z}p \oplus \mathbb{Z}q$$

where the first map is the differential d_2 and the second is d_1. We have $u \mapsto c-a+b$, $v \mapsto a+c-b$, and $a \mapsto p-q$, $b \mapsto p-q$, and $c \mapsto 0$. Via manual computation we deduce

$$\operatorname{im} d_1 = \langle p-q \rangle$$
$$\ker d_1 = \langle a-b, c\rangle, \ \operatorname{im} d_2 = \langle a-b+c, 2c\rangle$$
$$\ker d_2 = \langle 0 \rangle.$$

So with $\mathbb{Z}$-coefficients we have

$$H_0(X) = {}^{\langle p,q\rangle}\!/\!_{\langle p-q\rangle} \cong \langle p \rangle \cong \mathbb{Z}$$
$$H_1(X) = {}^{\langle a-b,c\rangle}\!/\!_{\langle a-b+c,2c\rangle} \cong {}^{\langle -c,c\rangle}\!/\!_{\langle 0,2c\rangle} \cong {}^{\langle c\rangle}\!/\!_{\langle 2c\rangle} \cong \mathbb{F}_2$$
$$H_2(X) = 0.$$

As the next step of our approach we consider the universal coefficient theorem:

$$H^n(X;\mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(H_n(X),\mathbb{Z}) \oplus \operatorname{Ext}(H_{n-1}(X),\mathbb{Z})$$

Using the properties of Ext from the lecture we compute

$$\operatorname{Ext}(H_2(X),\mathbb{Z}) \cong \mathcal{T}(\mathbb{F}_2) \otimes \mathbb{Z} \cong \mathbb{F}_2$$
$$\operatorname{Ext}(H_1(X),\mathbb{Z}) \cong 0$$
$$\operatorname{Ext}(H_0(X),\mathbb{Z}) \cong 0$$

In degree 1 this implies

$$H^1(X;\mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(H_1(X),\mathbb{Z})$$
$$\cong \operatorname{Hom}_{\mathbb{Z}}(\mathcal{F}(H_1(X)),\mathbb{Z}) \oplus \operatorname{Hom}_{\mathbb{Z}}(\mathcal{T}(H_1(X)),\mathbb{Z})$$
$$\cong \mathcal{F}(H_1(X)) = 0$$

where successively we use the equation for Ext, then we split the homology into free and torsion parts, and finally we use the canonical isomorphism that one can construct between a free group and its dual (basis elements map to their Kronecker delta), and the "lemma" that the dual of a torsion module vanishes; and in dimension 0

$$H^0(X;\mathbb{Z}) \cong \mathcal{F}(H_0(X)) \cong \mathbb{Z}$$

with the same arguments. To be able to compute cup products, the isomorphism type of $H^0(X;\mathbb{Z})$ is not enough: we need to find an explicit generator of $H^0(X;\mathbb{Z})$. From $H_0(X) = {}^{\langle p,q\rangle}/_{\langle p-q\rangle}$, we compute that the kernel of d^0 is given by $\phi_p + \phi_q$, and so $H^0(X) = \langle[\phi_p + \phi_q]\rangle$.

We now try to compute $H^2(X;\mathbb{Z})$ manually. We use the proposition that simplicial and singular cochain complexes are homotopy equivalent, so their cohomologies are naturally isomorphic. We consider a $\varphi\colon C_2^\Delta(X) \to \mathbb{Z}$ that is in $\ker d^2$. So by definition $\varphi \circ d_3 = 0$ as a map $C_3^\Delta(X) \to \mathbb{Z}$. Since $C_3^\Delta(X) = 0$ we deduce $\ker d^2 = \operatorname{Hom}_{\mathbb{Z}}\left(C_2^\Delta(X), \mathbb{Z}\right) =: (C_2^\Delta(X))^\vee$.

We have the equivalence

$$\alpha \in \operatorname{im} d^1 \Leftrightarrow \textit{there is a } \varphi \in (C_1^\Delta(X))^\vee \textit{ so that } \varphi \circ d_2 = \alpha$$

$$\begin{array}{ccc} \langle u,v\rangle & \xrightarrow{\alpha} & \mathbb{Z} \\ {\scriptstyle d_2}\downarrow & \nearrow_{\varphi} & \\ \langle a,b,c\rangle & & \end{array}$$

From algebra we know that φ is then uniquely determined by $\varphi(t)$ for $t = a, b, c$. The same is the case for α with u, v. Assuming that we have a pair (α, φ) as described above we must have some very specific relations. We have no choice than to write them down in a seemingly random fashion.

$$\alpha(u) = \varphi(d_2(u)) = \varphi(c - a + b) = \varphi(c) - \varphi(a) + \varphi(b)$$

$$\alpha(v) = \varphi(d_2(v)) = \varphi(a + c - b) = \varphi(a) + \varphi(c) - \varphi(b)$$

By ϕ_u we denote the map $C_2^\Delta(X) \to \mathbb{Z}$, $u \mapsto 1$, and $v \mapsto 0$ i.e. the identification we talked about earlier. Combining $\alpha = \varphi_u\phi_u + \varphi_v\phi_v = \varphi(a)(\phi_v - \phi_u) + \varphi(b)(\phi_u - \phi_v) + \varphi(c)(\phi_u + \phi_v)$, as we can choose φ as we wish we obtain

$$\operatorname{im} d^1 = \langle \phi_v - \phi_u, \phi_u + \phi_v\rangle.$$

Recall that $(C_2^\Delta)^\vee = \langle \phi_u, \phi_v\rangle$. So

$$H^2(X;\mathbb{Z}) = {}^{\ker d^2}/_{\operatorname{im} d^1} = {}^{\langle \phi_u, \phi_v\rangle}/_{\langle \phi_u + \phi_v, \phi_v - \phi_u\rangle}$$

$$\cong {}^{\langle \phi_u\rangle}/_{\langle \phi_u + \phi_u\rangle} \cong \mathbb{F}_2.$$

Thus we obtain a formula for the cohomology ring

$$H^\bullet(X;\mathbb{Z}) = \langle[\phi_p + \phi_q]\rangle \oplus {}^{\langle \phi_u\rangle}/_{\langle 2\cdot\phi_u\rangle}.$$

We have that $[\phi_p + \phi_q]$ has degree 0 and $[\phi_u]$ has degree 2 so then $[\phi_p + \phi_q] \smile [\phi_u]$ has degree 2. Hence $[\phi_p + \phi_q] \smile [\phi_u] = k[\phi_u]$ for $k \in \{0, 1\}$. By definition,

$$[\phi_p + \phi_q] \smile [\phi_u](u) = \phi_p(u|_{[v_0]})\phi_u(u) + \phi_q(u|_{[v_0]})\phi_u(u) = \phi_p(q)\cdot 1 + \phi_q(q)\cdot 1 = 1$$

where we use the notation $[v_0, \dots, v_j]$ for a standard j-simplex. It follows that $k = 0$.

An analogous computation implies that $[\phi_p + \phi_q] \smile [\phi_p + \phi_q] = [\phi_p + \phi_q]$ which makes sense because $[\phi_p]$ should be the unit element in the ring. We can thus write the formula

$$H^2(X;\mathbb{Z}) \cong {}^{\mathbb{Z}\langle y\rangle}/_{(y^2,2y)}$$

Co-homology of $X = \mathbb{R}P^2$ with coefficients in $\mathbb{F}_2$, $H_\bullet(X;\mathbb{F}_2)$. Analogously to $\mathbb{Z}$-Coefficients, the UCT yields a dual basis in degree $n = 1, 0$

$$\begin{cases} H^0(X;\mathbb{F}_2) = \langle \phi_p \rangle \\ H^1(X;\mathbb{F}_2) = \langle \phi_c \rangle \end{cases} \tag{6}$$

To get a basis of $H^2(X;\mathbb{F}_2)$ we once again work with the quotient of cocycles and coboundaries. Note that

$$Z^2_\Delta(X;\mathbb{F}_2) = C^2_\Delta(X;\mathbb{F}_2)$$
$$B^2_\Delta(X;\mathbb{F}_2) = \operatorname{Im}(d^1) = \langle \phi_u + \phi_v \rangle$$

Thus we can express $H^2_\Delta(X;\mathbb{F}_2)$ as

$$\left\{ H^2(K;\mathbb{F}_2) = \langle \phi_u, \phi_v \rangle / \langle \phi_u + \phi_v \rangle = \langle \phi_u \rangle \right. \tag{7}$$

By the same argumentation of degree, we must only search for non-zero products among co-chains of degree 1. To this end, let $\sigma = \alpha u + \beta v \in C_2^\Delta(X;\mathbb{F}_2)$ for vertices $e_0 \dots e_3$ such that $u = [e_0, e_2, e_3]$, $v = [e_1, e_2, e_3]$ and $\phi_i, \phi_j \in C^2_\Delta(X;\mathbb{F}_2)$ for $j, i \in \{a, b, c\}$. Investigating the image of a cup product on σ yields

$$\begin{aligned} \phi_i \smile \phi_j(\sigma) &= \alpha\phi_i(u|_{[e_0,e_2]})\phi_j(u|_{[e_2,e_3]}) + \beta\phi_i(v|_{[e_1,e_2]})\phi_j(v|_{[e_2,e_3]}) \\ &= \alpha\phi_i(b)\phi_j(c) + \beta\phi_i(a)\phi_j(c) \end{aligned}$$

Which yields a non-zero product, if

- $i = b$, $j = c$, which results in $\phi_b \smile \phi_c = \phi_u$, descending to $\phi_c \smile \phi_c = \phi_u$ in homology.
- $i = a$, $j = c$, which yields $\phi_a \smile \phi_c = \phi_v$, descending to $\phi_c \smile \phi_c = \phi_u$

Thus we can write the cohomology group of the real projective space $\mathbb{R}P^2$ with $\mathbb{F}_2$-Coefficitnes as $H_\bullet(X;\mathbb{F}_2) \cong \mathbb{F}_2[X]/\langle X^3 \rangle$, where $\deg(X) = 1$

PROBLEM 3

Maria Morariu

By Algebraic Topology 1, the homology groups with $\mathbb{Z}$-coefficients of M_g are: $H_0(M_g) \cong \mathbb{Z}$, $H_1(M_g) \cong \mathbb{Z}^{2g}$, $H_2(M_g) \cong \mathbb{Z}$, $H_k(M_g) \cong 0$ for $k > 2$. These are all free, so by the UCT for cohomology,
$H^k(M_g) \cong \mathrm{Hom}(H_k(M_g), \mathbb{Z})$. Hence, $H^0(M_g) \cong \mathbb{Z}$, $H^1(M_g) \cong \mathbb{Z}^{2g}$, $H^2(M_g) \cong \mathbb{Z}$ and $H^k(M_g) \cong 0$ for $k > 2$. Let $x_1, y_1, ..., x_g, y_g$ correspond to the generators of $H^1(M_g)$ and let z correspond to the generator of $H^2(M_g)$. Since all higher cohomology groups are zero, we can immediately deduce that the only pairs of non-zero elements whose cup product might be nonzero are $(x_i, y_j), (x_i, x_j), (y_i, y_j), (y_i, x_j)$ for $i, j \in \{1, ..., n\}$.
In the lecture, we have seen that $H^\bullet(T^2) \cong \mathbb{Z}\langle x, y\rangle/(xy + yx = x^2 = y^2 = 0)$.
Consider the map $q \colon M_g \to \bigvee_{i=1}^{g} T^2$ as shown in the given figure. q induces a homomorphism
$q^* \colon H^\bullet(\bigvee_{i_1}^{g} T^2) \to H^\bullet(M_g)$. We will use it to determine the other cup products.
Let us explicitly label the generators of the homology and cohomology groups. We visualize M_g as a polygon with $4g$ edges which are grouped into g tuples, each consisting of 4 consecutive edges labelled in counterclockwise order by $a_k, b_k, a_k^{-1}, b_k^{-1}$ for $1 \leq k \leq g$, by identifying edges according to the labelling (see figure for a depiction of the case $g = 2$). The a_i, b_i are in fact cycles, representing generators of $H_1(M_g)$. Also, let σ be a cycle representing the generator of $H_2(M_g)$. Let us denote a'_i, b'_i the respective cycles corresponding to generators of $H_1(\bigvee_{i_1}^{g} T^2) \cong \bigoplus_{i=1}^{g} H_1(T^2)$, meaning $a'_i = q_c(a_i)$ and $b'_i = q_c(b_i)$, where by q_c we denote the chain map induced by q. Let $\sigma'_1, ..., \sigma'_g$ be cycles generating $H_2(T^2)$ for each of the $H_2(T^2)$ in $H_2(\bigvee_{i_1}^{g} T^2) \cong \bigoplus_{i=1}^{g} H_2(T^2)$. For each i, let $\varphi'_i \in C^1(\bigvee_{i_1}^{g} T^2)$ be the dual of a'_i and $\psi'_i \in C^1(\bigvee_{i_1}^{g} T^2)$ be the dual of b'_i. Also, let $\eta'_i \in C^2(\bigvee_{i_1}^{g} T^2)$ be the dual of σ'_i. Set

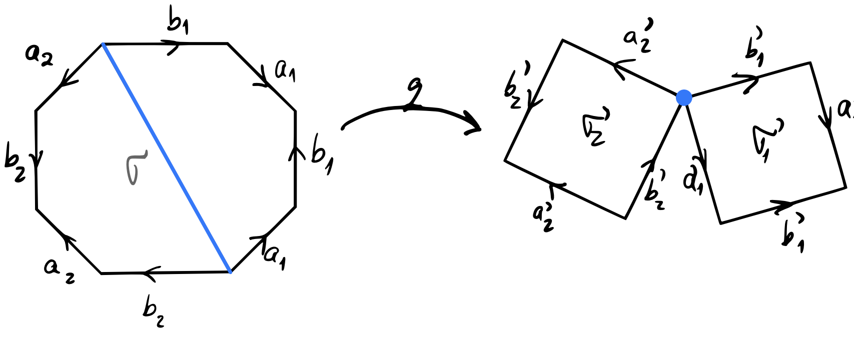


FIGURE 1. Visualization of M_g for $g = 2$

$\varphi_i := \varphi'_i \circ q$ and $\psi_i := \psi'_i \circ q$. Then $q^*([\varphi'_i]) = [\varphi'_i \circ q] = [\varphi_i]$. We observe that for each i, $\varphi_i(a_j) = \varphi'_i(a'_j) = \delta_{i,j}$ and $\varphi_i(b_j) = \varphi'_i(b'_j) = 0$, so φ_i is the dual map of a_i. Hence, $[\varphi_i]$ is a generator of $H^1(M_g)$, let it correspond to x_i. Similarly, ψ_i is the dual map of b_i and we let it correspond to y_i. Let $\eta \in C^2(M_g)$ be the dual map of $\sigma \in C_2(M_g)$. $[\eta]$ generates $H^2(M_g)$, let it correspond to z. Note that $q(\sigma) = \sigma'_1 + ... + \sigma'_g$, so for each i, $\eta'_i \circ q(\sigma) = \sum_{j=1}^{g} \eta'_j(\sigma'_i) = 1$, so $\eta'_i \circ q = \eta$.
The computation done in the lecture for the cohomology ring of the torus, together with the explicit description of $H^\bullet(T^2 \vee ... \vee T^2)$, shows $[\varphi'_i] \smile [\varphi'_i] = [\psi'_i] \smile [\psi'_i] = 0$ for each i. Therefore, using naturality of the cup product, we obtain $[\varphi_i] \smile [\varphi_i] = q^*([\varphi'_i] \smile [\varphi'_i]) = 0$ and $[\psi_i] \smile [\psi_i] = q^*([\psi'_i] \smile [\psi'_i]) = 0$ for each i.
Also by the computation in the lecture, we have $[\varphi'_i] \smile [\psi'_i] = [\eta'_i]$. Hence,
$[\varphi_i] \smile [\psi_i] = q^*([\varphi'_i] \smile [\psi'_i]) = q^*([\eta'_i]) = [\eta'_i \circ q] = [\eta]$. Similarly, we obtain for any j $[\psi_j] \smile [\varphi_j] = -[\eta]$. This means that for any $i, j : [\varphi_i] \smile [\psi_i] + [\psi_j] \smile [\varphi_j] = 0$.
Now for $i \neq j$, we have $[\varphi_i] \smile [\varphi_j] = q^*([\varphi'_i] \smile [\varphi'_j]) = q^*(0) = 0$. Similarly, $[\psi_i] \smile [\psi_j] = 0$.
Therefore,

$$\begin{aligned} H^\bullet(M_g) \cong \mathbb{Z}\langle x_1, y_1, ..., x_g, y_g\rangle / \\ (x_i y_i + y_j x_j = 0 \forall i, j,\ x_i y_j = y_i x_j = 0 \forall i \neq j,\ x_i x_j = y_i y_j = 0 \forall i, j), \end{aligned}$$

where all x_i and y_i have degree 1.

Problem 4

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Calculate the cohomology ring,

$$H^\bullet(\mathbb{CP}^2;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^3)$$

with x the generator of degree 2 and

$$H^\bullet(\mathbb{S}^2;\mathbb{Z}) \oplus H^\bullet(\mathbb{S}^4;\mathbb{Z}) \cong \mathbb{Z}[y]/(y^2) \oplus \mathbb{Z}[z]/(z^2)$$

with y the generator of degree 2 and z the generator of degree 4.

The cohomology and its reduced version only differ in degree 0. We consider the generator of degree 2. The degree 2 generator of $H^\bullet(\mathbb{S}^2;\mathbb{Z}) \oplus H^\bullet(\mathbb{S}^4;\mathbb{Z})$ is y and it is the generator of degree 2 of the reduced cohomology $\widetilde{H}^\bullet(\mathbb{S}^2;\mathbb{Z}) \oplus \widetilde{H}^\bullet(\mathbb{S}^4;\mathbb{Z})$, and can be identified as the generator of degree 2 of $\widetilde{H}^\bullet(\mathbb{S}^2 \vee \mathbb{S}^4;\mathbb{Z})$ by $\widetilde{H}^\bullet(\mathbb{S}^2 \vee \mathbb{S}^4;\mathbb{Z}) \cong \widetilde{H}^\bullet(\mathbb{S}^2;\mathbb{Z}) \oplus \widetilde{H}^\bullet(\mathbb{S}^4;\mathbb{Z})$, and again is the generator of degree 2 of $H^\bullet(\mathbb{S}^2 \vee \mathbb{S}^4;\mathbb{Z})$. We have $y^2 = 0$, but x is the generator of degree 2 of $H^2(\mathbb{CP}^2;\mathbb{Z})$ and x^2 is the generator of $H^4(\mathbb{CP}^2;\mathbb{Z})$ which is nonzero. We have $H^\bullet(\mathbb{CP}^2;\mathbb{Z}) \ncong H^\bullet(\mathbb{S}^2 \vee \mathbb{S}^4;\mathbb{Z})$. The two spaces have different cohomology ring, they are not homotopic equivalent.

Remark 1:

$$\widetilde{H}^\bullet(X \vee Y;R) \cong \widetilde{H}^\bullet(X;R) \oplus \widetilde{H}^\bullet(Y;R)$$

can be proved by typical method as we seen in AlgebraicTopology I:

$$\to \widetilde{H}^{i-1}(N_X \cap N_Y;R) \to \widetilde{H}^i(X \vee Y;R) \to \widetilde{H}^i(X \cup N_Y;R) \oplus \widetilde{H}^i(Y \cup N_X;R) \to \widetilde{H}^i(N_X \cap N_Y;R) \to$$

for $i \geq 1$ and

$$0 \to \widetilde{H}^0(X \vee Y;R) \to \widetilde{H}^0(X \cup N_Y;R) \oplus \widetilde{H}^0(Y \cup N_X;R) \to \widetilde{H}^0(N_X \cap N_Y;R) \to$$

with N_X, N_Y be open neighbourhoods of the connecting point deformation retracting to it. Thus

$$\widetilde{H}^i(X \vee Y;R) \cong \widetilde{H}^i(X;R) \oplus \widetilde{H}^i(Y;R)$$

for $i \geq 0$.

$$\widetilde{H}^\bullet(X \vee Y;R) := \bigoplus_{i=0}^{\infty} \widetilde{H}^i(X \vee Y;R) \cong \bigoplus_{i=0}^{\infty} (\widetilde{H}^i(X;R) \oplus \widetilde{H}^i(Y;R))$$

$$\cong (\bigoplus_{i=0}^{\infty} \widetilde{H}^i(X;R)) \oplus (\bigoplus_{i=0}^{\infty} \widetilde{H}^i(Y;R)) =: \widetilde{H}^\bullet(X;R) \oplus \widetilde{H}^\bullet(Y;R)$$

Concretely, $H^\bullet(X \vee Y;R)$ is the subalgebra of $H^\bullet(X;R) \times H^\bullet(Y;R)$ containing in degree 0 only those (φ,ψ) with $\varphi(x_0) = \psi(y_0)$. While $\widetilde{H}^\bullet(X \vee Y;R)$ contains exactly $(\varphi,\psi) \in \widetilde{H}^\bullet(X;R) \times \widetilde{H}^\bullet(Y;R)$.

Remark 2:

$$H^\bullet(X \vee Y;R) \cong H^\bullet(X;R) \oplus H^\bullet(Y;R)$$

is not correct for the 0-th cohomology. For instance, the sum of cohomology ring of $\mathbb{S}^2$ and $\mathbb{S}^4$ is of rank 2 in degree 0 whereas the cohomology ring of the wedge is of rank 1 in degree 0.

Problem 5

5 a). Let $U_1, \dots, U_k$ be an open cover of X such that each inclusion $U_i \hookrightarrow X$ is nullhomotopic. For $1 \le i \le k$, let $\alpha_i \in H^{n_i}(X)$ with $n_i \ge 1$ be given. We need to show that $\alpha_1 \smile \cdots \smile \alpha_k = 0$.
Since each cohomology restriction map $H^{n_i}(X;\mathbb{F}) \to H^{n_i}(U_i;\mathbb{F})$ is trivial, the classes α_i lift to classes β_i in the relative cohomology groups $H^{n_i}(X, U_i;\mathbb{F})$. From the naturality of the relative cup product, it follows that $\alpha_1 \smile \cdots \smile \alpha_k$ is the image of $\beta_1 \smile \cdots \smile \beta_k$, and the latter is an element of

$$H^*(X, \bigcup_i U_i; F) = H^k(X, X; F) = 0.$$

Hence the product $\alpha_1 \smile \cdots \smile \alpha_k$ equals zero, as desired.

5 b). By Problem 5.a), the Lusternik-Schnirelmann category of the sphere S^n is at least two. The corresponding open cover is given by complements to north and south poles $U_\pm \coloneqq \big\{S^n \setminus \{(0, \dots, 0, \pm 1)\}\big\}$

5 c). By Problem 5.a), the Lusternik-Schnirelmann category of the projective space $\mathbb{C}P^n$ is at least $n+1$. The corresponding open cover is given the canonical charts $U_i \coloneqq \{[z_0 : \cdots : z_n] \mid z_i \neq 0\}$.

5 d). Since there are n classes in $H^1(T^n;\mathbb{F})$ whose cup product is nonzero, Problem 5.a) implies that T^n has Lusternik-Schnirelmann category at least $n+1$.
One can construct an explicit open covering of T^n with $n+1$ open sets as follows. Let $p\colon \mathbb{R}^n \to T^n$ be the usual universal covering projection sending $(t_1, \dots, t_n)$ to $(e^{2\pi i t_1}, \dots, e^{2\pi i t_n})$, and let $a_0, \dots, a_n$ be distinct points in the half-open interval $[0, 1)$, so that the points $z_k = e^{2\pi i a_k} \in S^1$ are distinct. Now let $W_k \subset \mathbb{R}^n$ be the set of all points such that $a_k < t_i < a_k + 1$ for all i, and take $V_k \subset T^n$ to be the image of W_k under p. By construction each set V_k is contractible. A point of T^n will lie in $T^n - V_k$ if and only if at least one of its coordinates is equal to z_k. The intersection of the sets $T^n - V_k$ will consist of all points $(b_1, \dots, b_n)$ such that for each k, there is some j for which $b_j = z_k$. Since there are $n+1$ values of z_k and only n coordinates b_j, this is impossible. Therefore $\bigcap_k (T^n - V_k) = \varnothing$, so that $T^n = \bigcup_k V_k$.